Lecture 2

Recap: Cardinality of Infinite Sets

- We proved the following are countably infinite: $\{0,1\}^*, \mathbb{Z}, \mathbb{Q}, \mathbb{N}^2, \mathbb{N}^3, \cdots$
- We concluded that a set B is countably infinite if you can enumerate the elements of $B = \{x_1, x_2, x_3, \cdots\}$.
- The above is equivalent to finding a one-one function $f : B \to \mathbb{N}$.

This Lecture

1. Proof of $|\mathbb{R}| > |\mathbb{N}|$
2. Computational Problems
    a. Simplifying Restrictions
    b. Computational Problems as languages and strings.
    c. Cardinality of languages.
3. Our first impossibility result.

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Definitions

- **Definition:** A set A is uncountable if it is not countable (i.e. $|A| > |\mathbb{N}|$).

- **Definition:** A set A is uncountable if there is no one-one function from A to $\mathbb{N}$.

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To show $|\mathbb{R}| > |\mathbb{N}|$, it is sufficient to show $|I| > |\mathbb{N}|$, where $I = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$.

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Key Questions in This Course

- What problems can you solve with a computer?
  - **Computability Theory**
- Why are some problems harder to solve than others?
  - **Complexity Theory**
- What are some tools we will use to formalize our answers to these questions?
  - **Discrete Math**

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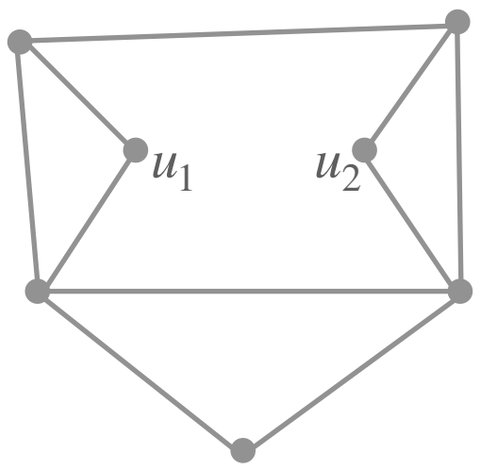

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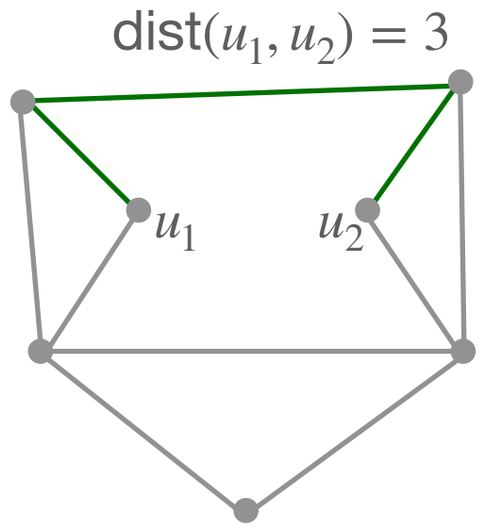

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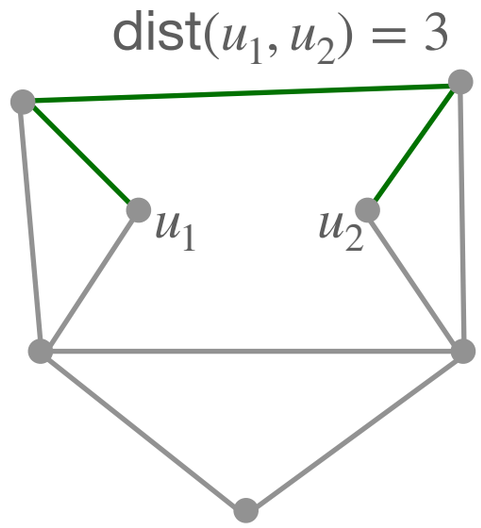


In other words, a computational problem is a relation from the set of valid inputs to the set of outputs.

Computational Problem

A computational problem is relation from the set of valid inputs to the set of outputs. Too general!

Simplifying Restriction 1: We represent inputs as binary strings.

Simplifying Restriction 2: We will consider only decision problems. That is the output is YES/NO.

Strings

- We use $\{0,1\}$ to denote the two elements of the binary alphabet.
- $\{0,1\}^n$ is denoted as the set of all binary strings of length n.
- The unique string in $\{0,1\}^0$ is the empty string, denoted ϵ.

  (common in many programming languages as “”)
- Let x be a binary string, then $|x|$ denotes the length of the string.
- We use $\{0,1\}^*$ to denote the collection of all strings of finite length. That is, $\{0,1\}^* = \bigcup_{n \in \mathbb{N}} \{0,1\}^n$.
- All the above definitions can be applied to any alphabet Σ. An alphabet is just a finite collection of symbols.

Back to Restriction 1

Simplifying Restriction 1: We represent inputs as binary strings.

Why is this reasonable?

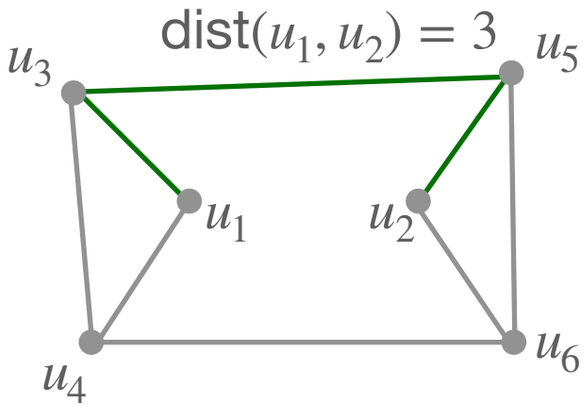
Answer:

- We can encode any object with a finite description with a binary string.

Proposition: For every finite set $\mathcal{X}$ with k elements, there is a one-to-one encoding function $h : \mathcal{X} \to \{0,1\}^{\lceil \log k \rceil}$.

Restriction 1: Example

Example: Given as input a graph G, and pair of vertices u_1, u_2 output the distance between u_1, u_2.



Graph G

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0	0	1	1	0	0
u_2	0	0	0	0	1	1
u_3	1	0	0	1	1	0
u_4	1	0	1	0	0	0
u_5	0	1	1	0	0	1
u_6	0	1	0	0	1	0

The binary encoding of G denoted, $\langle G \rangle \in \{0,1\}^{n^2}$ obtained by unraveling the adjacency matrix.

For any object B, we will denote its binary encoding by $\langle B \rangle$.

Restriction 2

Simplifying Restriction 2: We will consider only decision problems. That is the output is YES/NO.

Objection: There are many natural problems that interest us for which the desired output is not simply a Yes or No answer.

(Partial) Answer:

In many cases, even if the problem we are interested in is not a decision problem, we can formulate a single or multiple related decision problems that "capture the essence of" the problem.

Example: Given as input a graph G and vertices u_1, u_2, is there a path of length at most 3 between u_1, u_2?

Restriction 2

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Functions and Languages

- Given Restriction 1 and Restriction 2, a *decision problem* where all the inputs are binary strings of length n can be described with a *Boolean function:* $f : \{0,1\}^n \to \{0,1\}$.

Example: Given as input a graph $G = (V, E)$ with $|V| = n$ and vertices u_1, u_2, is there a path of length at most 3 between u_1, u_2?

Can be described as a Boolean function $f : \{0,1\}^{n^2+2\lceil \log n \rceil} \to \{0,1\}$.

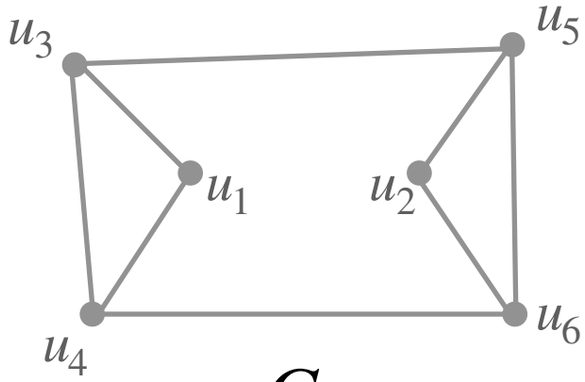
Functions and Languages

- In this course, we will consider ways to solve a computational problems using a computational model.
- But a “computer code” doesn’t take a fixed length input!
- So, we represent decision problems where input can be of any length as:
  - Family of Boolean Functions: $\cup_{n \geq 0} f_n$ where $f_n : \{0,1\}^n \rightarrow \{0,1\}$.
  - As languages: a language $L \subseteq \{0,1\}^*$ is a set of all valid inputs on which we return YES.

Functions and Languages

Example: Given a graph G, output whether or not it contains a cycle of length 3.

Corresponding Language: $L = \{\langle G \rangle \mid G \text{ has a cycle of length } 3\}$.



G

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	0	0	1	1	0	0
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u_6	0	1	0	0	1	0

$\langle G \rangle$

$\langle G \rangle \in L$

Functions and Languages

Equivalence of the two representations:

- Given a language L_D representing a decision problem D. Then, all strings of length n in L_D can be represented by a Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ such that $f(x) = 1 \iff x \in L_D$.

- Similarly, any family of Boolean functions $\cup_{n \geq 0} f_n$ where, $f_n : \{0,1\}^n \rightarrow \{0,1\}$ corresponds to the language $L = \cup_{n \geq 0} f^{-1}(1)$.

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Review: Countability

- Claim 1: For a fixed $n \geq 0$, the set $\{0,1\}^n$ is finite.
- Claim 2: The set $\{0,1\}^*$ is countably infinite.

Review: Countability

- Claim 1: For a fixed $n \geq 0$, the set $\{0,1\}^n$ is finite.
- Claim 2: The set $\{0,1\}^*$ is countably infinite.

Note: any language $L \subseteq \{0,1\}^*$, thus it is countable.

Cardinality of Languages

- **Claim:** The set of all languages L is uncountable.
- Proof by Diagonalization! (We will see more of this technique.)

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Can all computational problems be solved?

- What does it mean for a computational problems to be solved?
- Let's say it means that we can write a python code to solve it.
- Alternatively, a language L can be solved if and only if there is a python code P_L such that $P_L(x) = 1 \iff x \in L$.

- **Claim:** There exists a language L for which there is no python code P such that $P(x) = 1 \iff x \in L$.

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Some Thoughts on the Proof

- There is nothing special about Python. You can replace it with a programming language or machine model of your choice.
- The key step: any algorithm has a finite description and we can therefore encode them as strings.
- Objection 1: The proof is non-constructive. What are the problems that cannot be solved by a python code? Are they interesting?
- Objection 2: The proof says: for every computational model there is a problem that cannot be solved by it. Is there a problem that cannot be solved by all computational models simultaneously?

Summary

1. **Computational Problems:** is a *task* which for each *valid input* produces one or more *outputs.*
2. **Two Simplifying Restrictions:**
   - Inputs are binary strings
   - Outputs are 0/1 or YES/NO.
3. **Decision Problems = Languages:**
   $L \subseteq \{0,1\}^*, x \in L \iff$ answer on x is YES.
4. **Cardinality of Languages:** The set $\{0,1\}^*$ is countable but, the set of all languages is uncountable.
5. **Our First Impossibility Result:** The set of all Python programs is countable but the set of all languages is uncountable. So, there are languages which cannot be solved by any Python program.