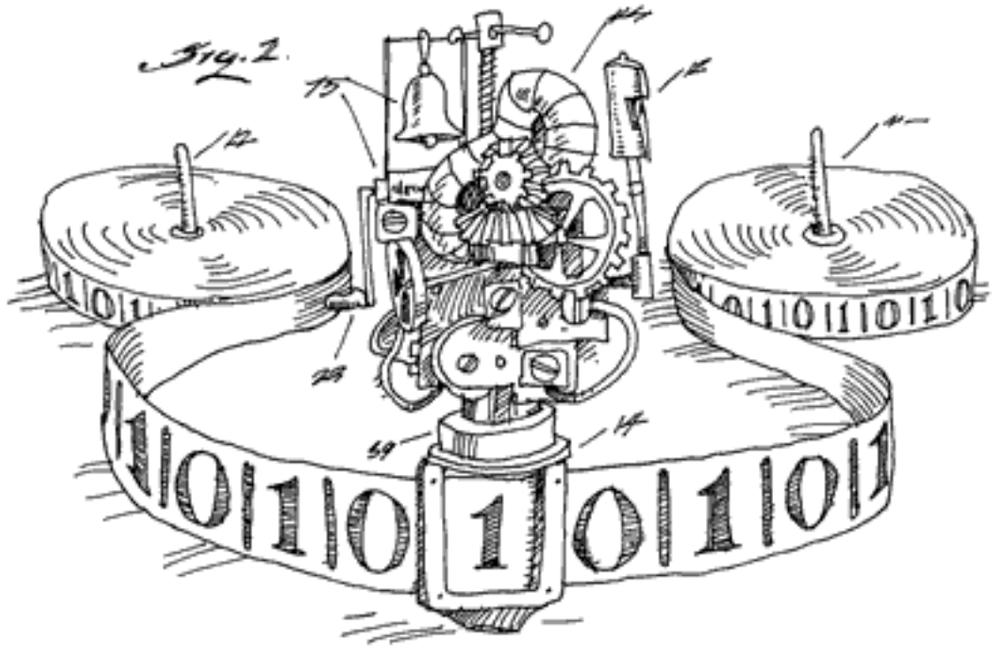
CSC4010: Introduction to Theoretical Computer Science



Lecture 3

Logistics

- Is everyone getting Blackboard emails?

- Part of Problem Set 1 is out, contains three problems. We will discuss in tutorial. You should attempt this, alone or in groups.

- Course webpage is also updated with topics to read from references. Please make sure to have a look.

Recap

1. **Computational Problems:** is a *task* which for each *valid input* produces one or more *outputs.*

2. **Two Simplifying Restrictions:**
   - Inputs are binary strings
   - Outputs are 0/1 or YES/NO.

3. **Decision Problems = Languages:**
   $L \subseteq \{0,1\}^*, x \in L \iff$ answer on x is YES.

4. **Cardinality of Languages:** The set $\{0,1\}^*$ is countable but, the set of all languages: $\mathbb{L} = \{L : L \subseteq \{0,1\}^*\},$ is uncountable.

Recap

Fact 1: $\{0,1\}^*$ is countable.

$\{0,1\}^* = \{x_1, x_2, \ldots\}$

Claim: The set of all languages: $\mathbb{L} = \{L : L \subseteq \{0,1\}^*\}$, is uncountable.

Proof: (By contradiction) $\mathbb{L}$ is countable, $\mathbb{L} = \{L_1, L_2, \ldots\}$.

	x_1	x_2	x_3	x_4
L_1	0	1	1	…
L_2	0	1	0	
L_3	1	1	0	
⋮				

$= M(L_j, x_i) = 1 \iff x_i \in L_j$ and o/w.

$L_{diag} = \{x_i \mid x_i \notin L_i\}$

$L_{diag} = \{x_1, x_3 \ldots\}$

$L_{diag} \neq L_k$ as it disagrees w/ L_k at x_k

$L_{diag} \notin \{L_1, L_2, \ldots\} \rightarrow (2)$

$L_{diag} \subseteq \{0,1\}^*$

$L_{diag} \in \mathbb{L} \rightarrow (3)$

(2) → (3) ⇒ contradiction

Recap

Observation: Python program P, it $\langle P \rangle$ is also finite.

Claim: There exists a language L for which there is no python code P such that $P(x) = 1 \iff x \in L$.

$\mathbb{P} = \{ \langle P \rangle \mid P \text{ is a python program} \} \subseteq \{0,1\}^*$ (countable)

$|\mathbb{P}| \leq |\mathbb{N}|$ → ①

Proof: (Contradiction) For every language $L \in \mathbb{L}$, $\exists$ Python code P_L s.t. $P_L(x) = 1 \iff x \in L$.

$f : \mathbb{L} \to \mathbb{P}$. $f(L) = \langle P_L \rangle$ (one-one)

⇓

as python prog corresponding to each language is distinct.

$|\mathbb{L}| \leq |\mathbb{P}|$ (f is one-one)

$|\mathbb{L}| > |\mathbb{N}|$ (uncountable)

$|\mathbb{L}| > |\mathbb{P}|$ ② $|\mathbb{P}| = |\mathbb{N}|$

Some Thoughts on the Proof

- There is nothing special about Python. You can replace it with a programming language or machine model of your choice.
- The key step: any algorithm has a finite description and we can therefore encode them as strings.
- Objection 1: The proof is non-constructive. What are the problems that cannot be solved by a python code? Are they interesting?
- Objection 2: The proof says: for every computational model there is a problem that cannot be solved by it. Is there a problem that cannot be solved by all computational models simultaneously?

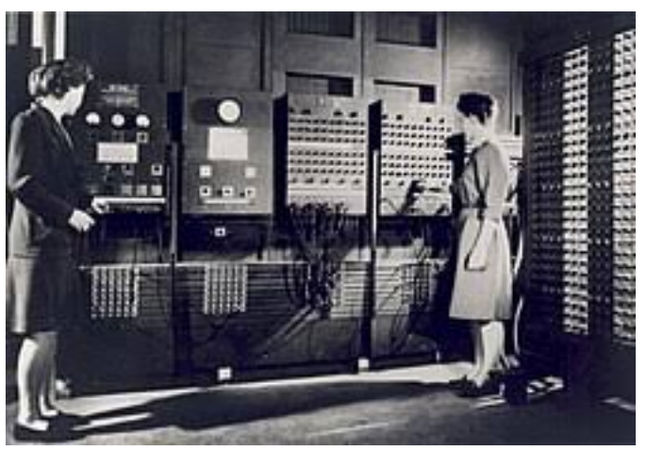
Our Hero for This Lecture

Alan Turing (1912-1954)
Founder of Computer Science
Mathematician, Philosopher,
Codebreaker,
World Class Marathoner

- Designed the Turing Machine, foundation of the modern theory of computation and computability.
- Answered the Decidability Question.
- Contributed to breaking the German Enigma code during WWII.
- Designed in theory, one of the first first computers (ACE) and the first programming languages (ACI).
- His best marathon time of 2:46:03 was only 11 minutes slower than the 1948 Olympic champion.

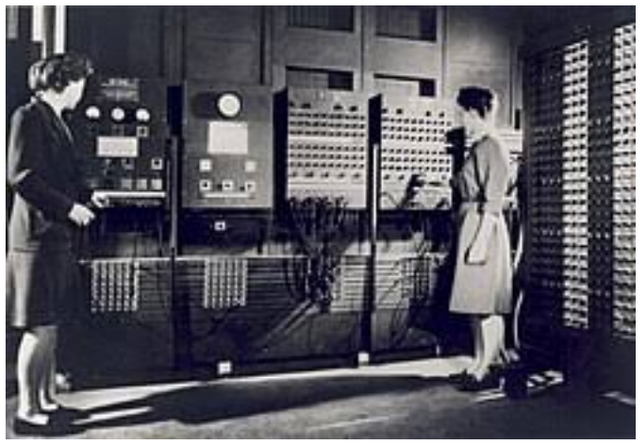
Computation

- Computers: modern technology?
- But computation is ancient.
- Can be performed by:
  - A human being with a pen and a pencil.
  - A smart-phone.
  - A supercomputer.
- We want a mathematical model that describes all of these and transcends any one technology.



Computation

- Note: Humans do the same computation that (smartphones/ laptops do)
  - Less quickly and less reliably
- Consequence: we can do computation without understanding electronics.
- Computation is a familiar, everyday, human act.



Turing Machine

- We need a precise mathematical meaning of the term “algorithm”.
- The model should not depend on a particular program, programming language, or technology. Should capture the essence of computation.
- Turing Machines captures computation using a finite set of simple rules that allow for one small operation at a time.

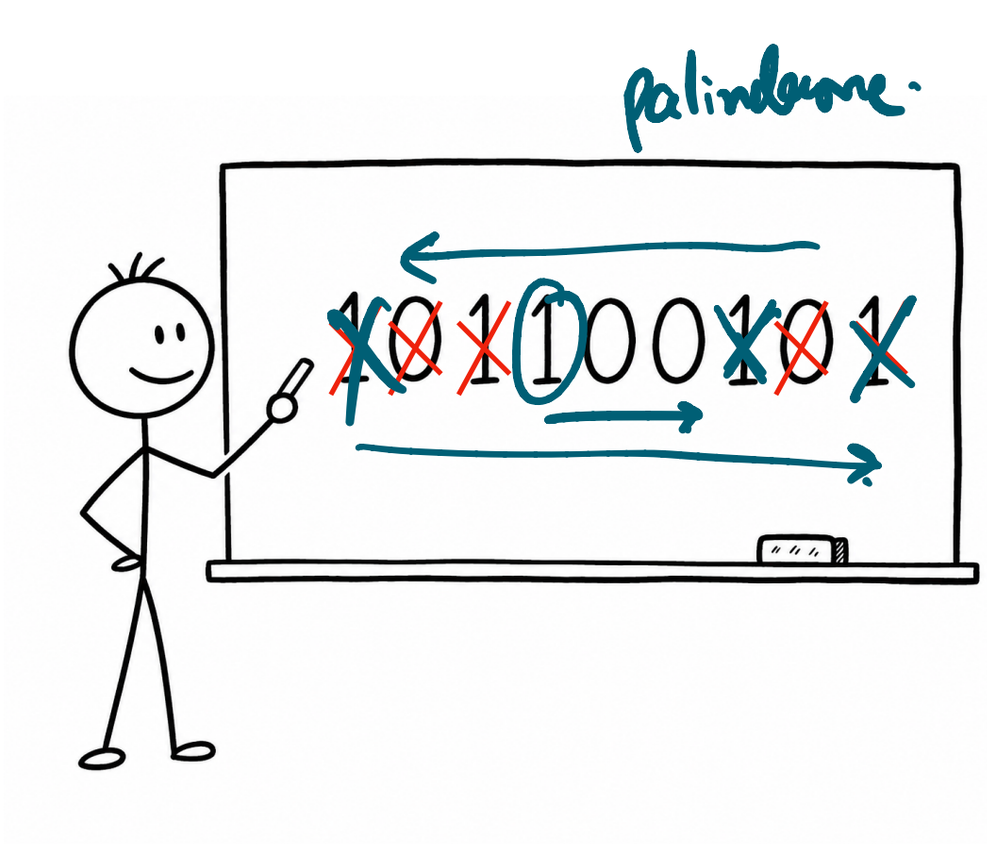
010 is a palindrome, 100 is not a palindrome.

Example: Palindromes

- Suppose a long string of bits is written on a blackboard.
- Our job: Figure out whether the string is a “palindrome,” i.e., whether it is the same forwards and backwards.
- What should we do?

Example: Palindromes

- Idea: Compare and cross off the first and last symbols.
- Repeat until there is a mismatch or everything is crossed off.



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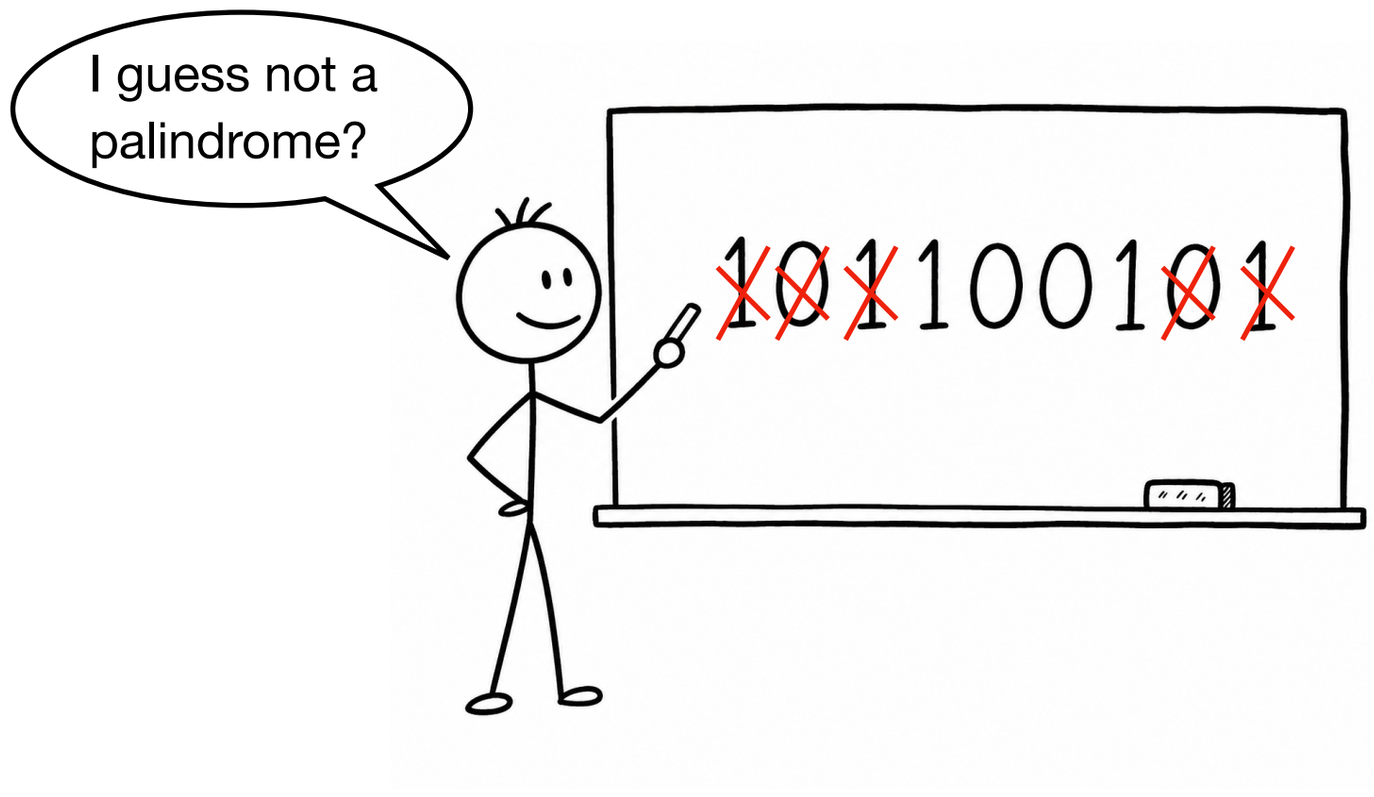


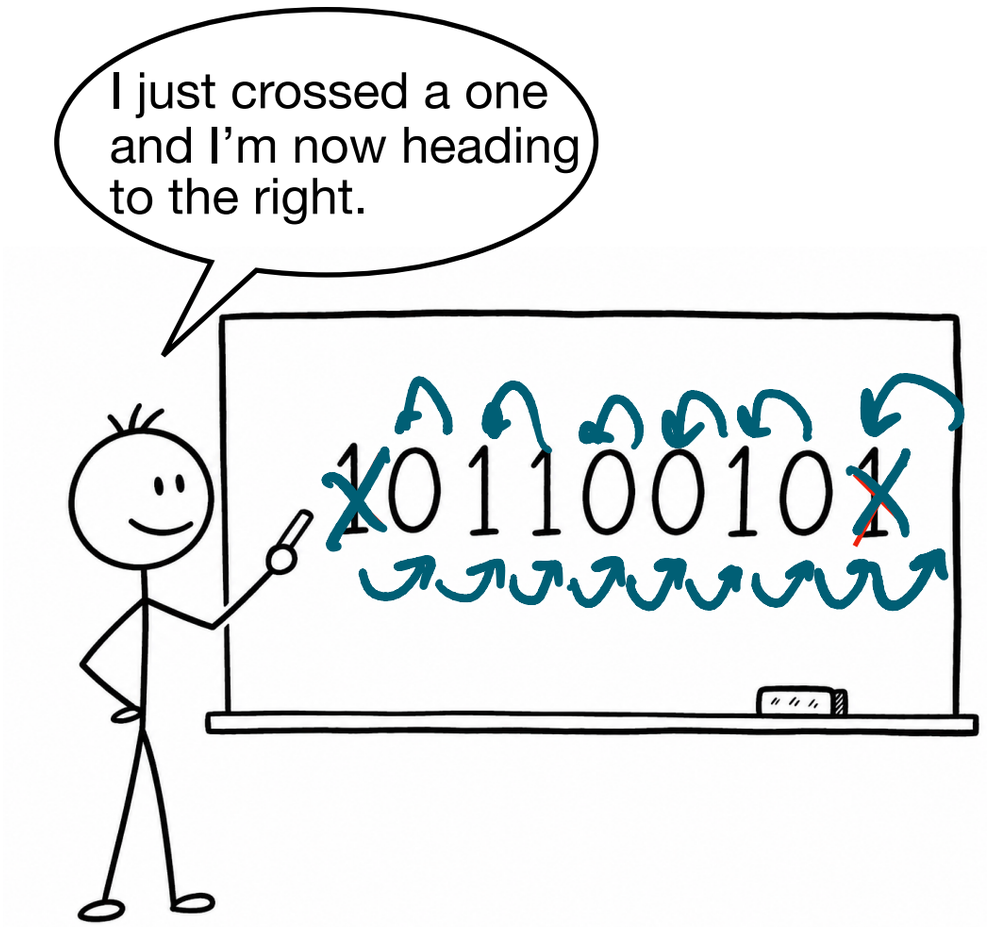
Local Decisions

- In each step, how do we know what to do next?

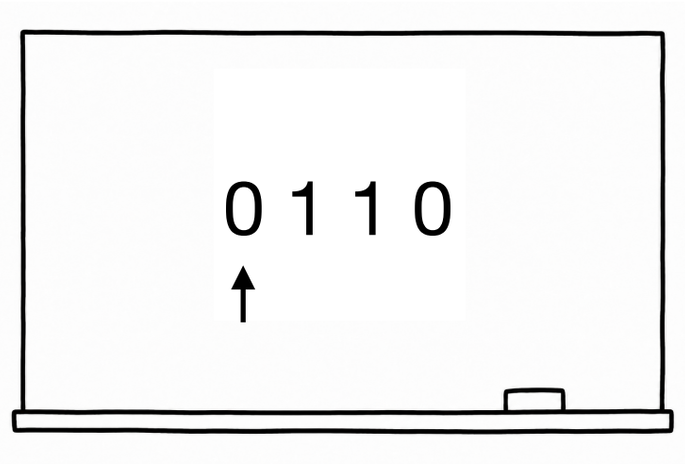
  1. We keep track of some information (“state”) in our head.

  2. We look at the local contents of the blackboard. (One symbol is sufficient.)

  3. We can describe the algorithm using a “state diagram”



State Diagram



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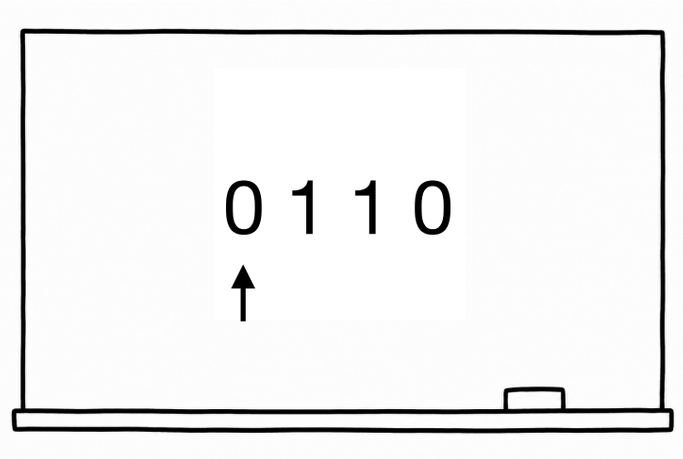

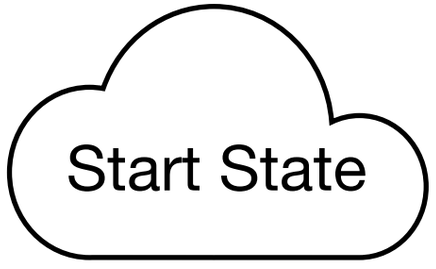

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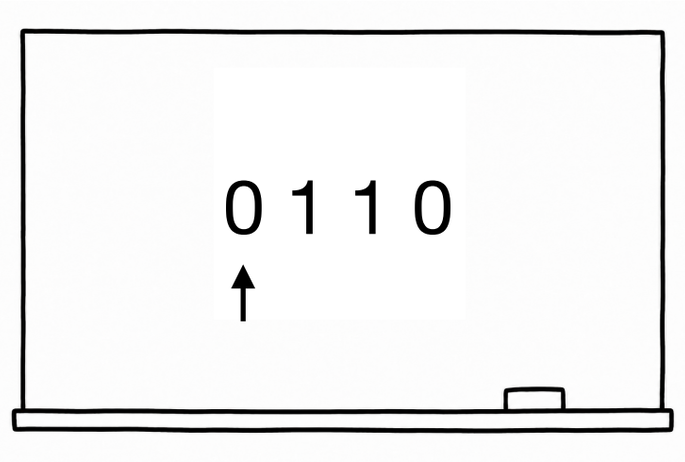

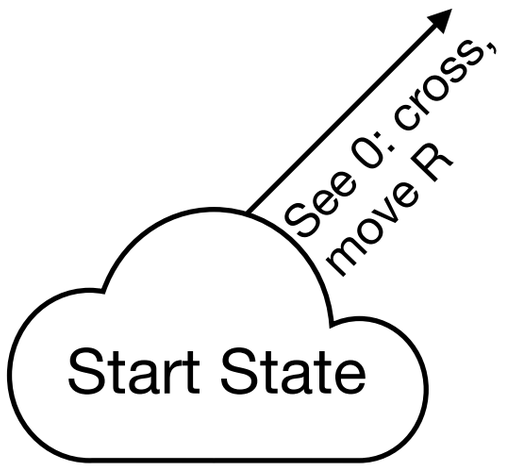

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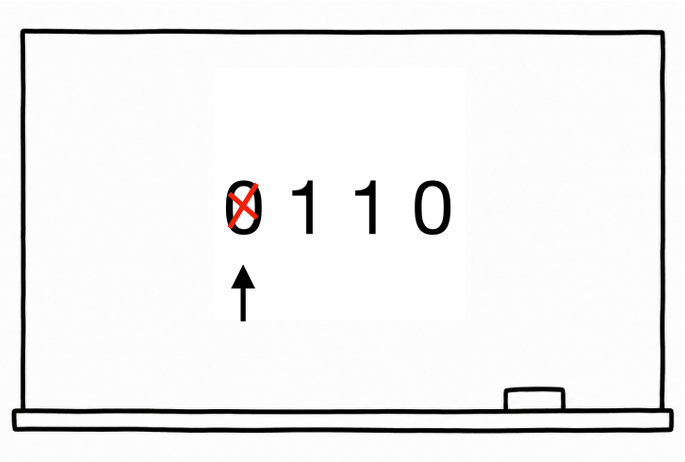

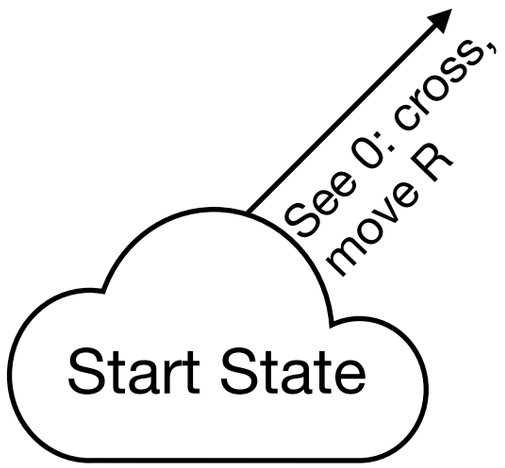

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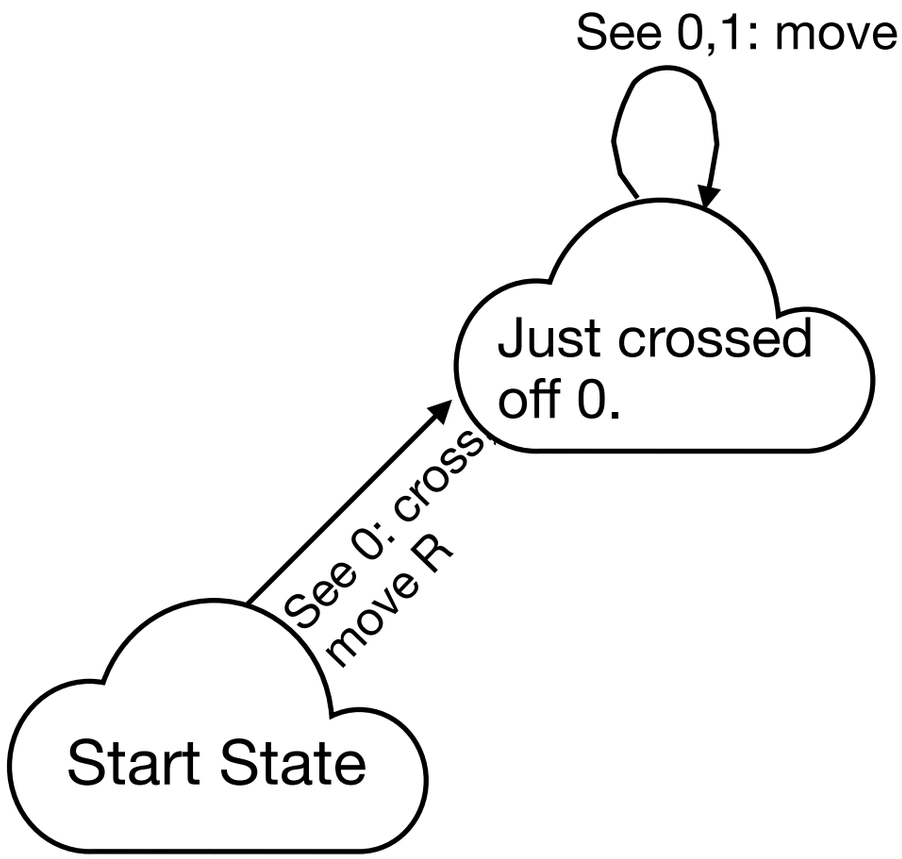

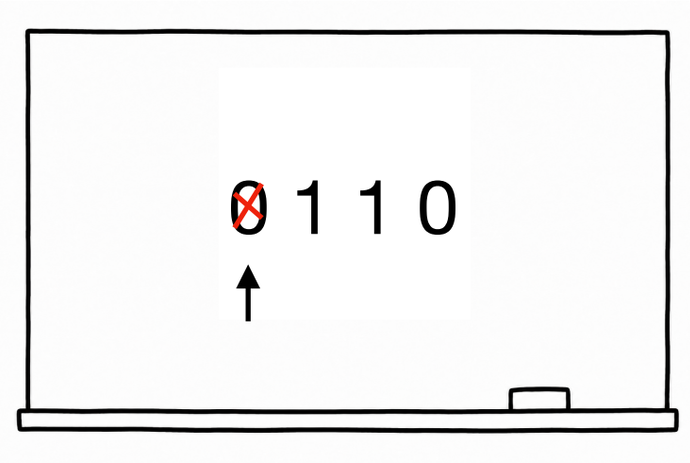

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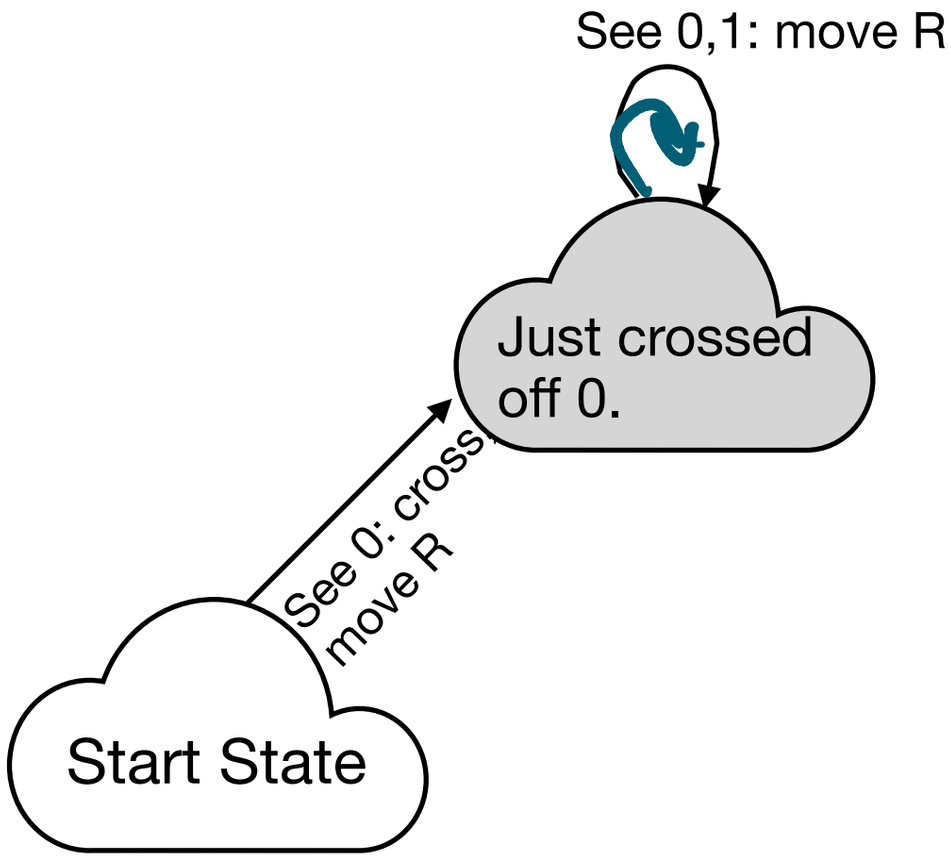

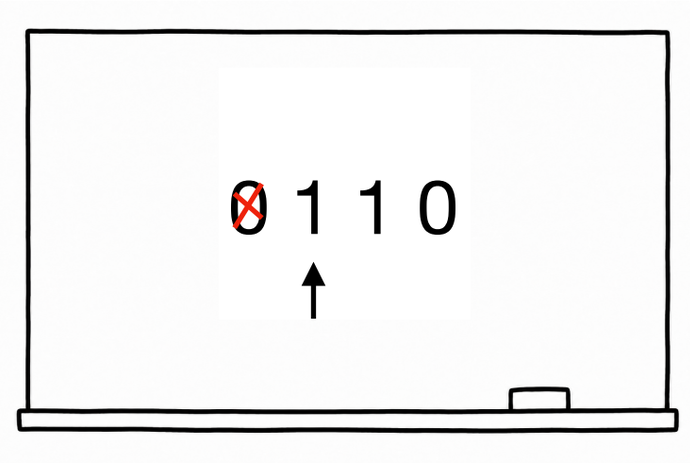

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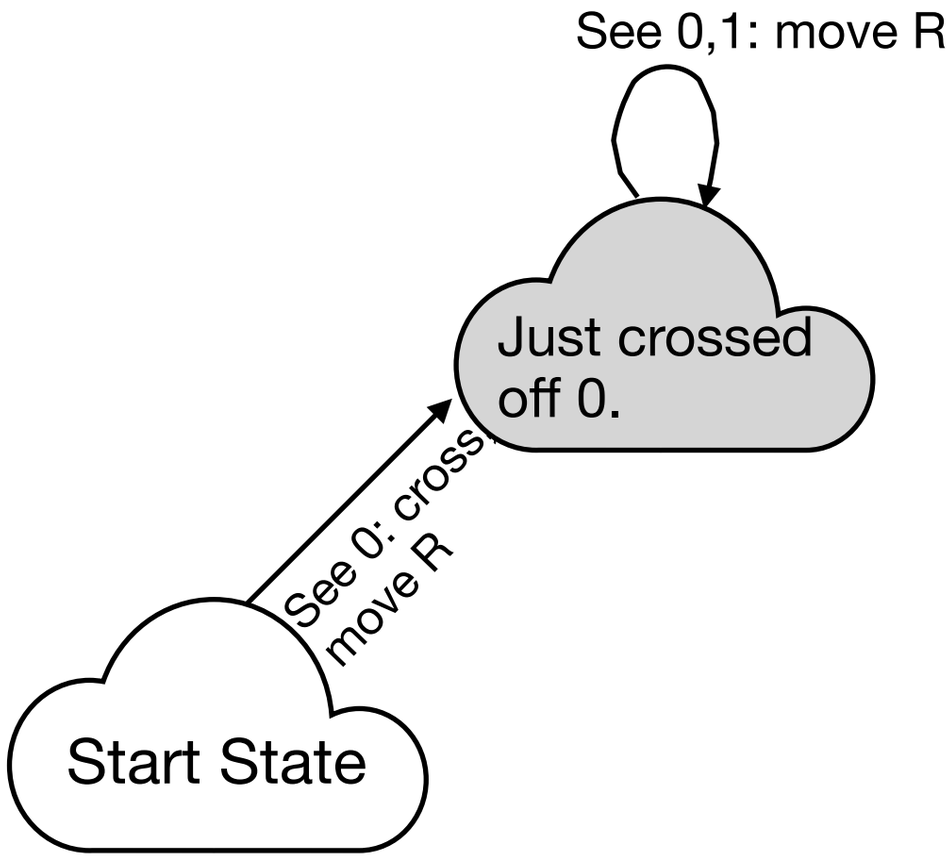

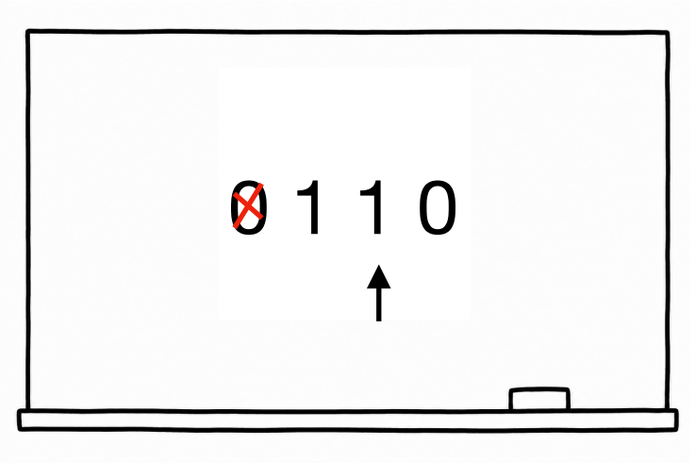

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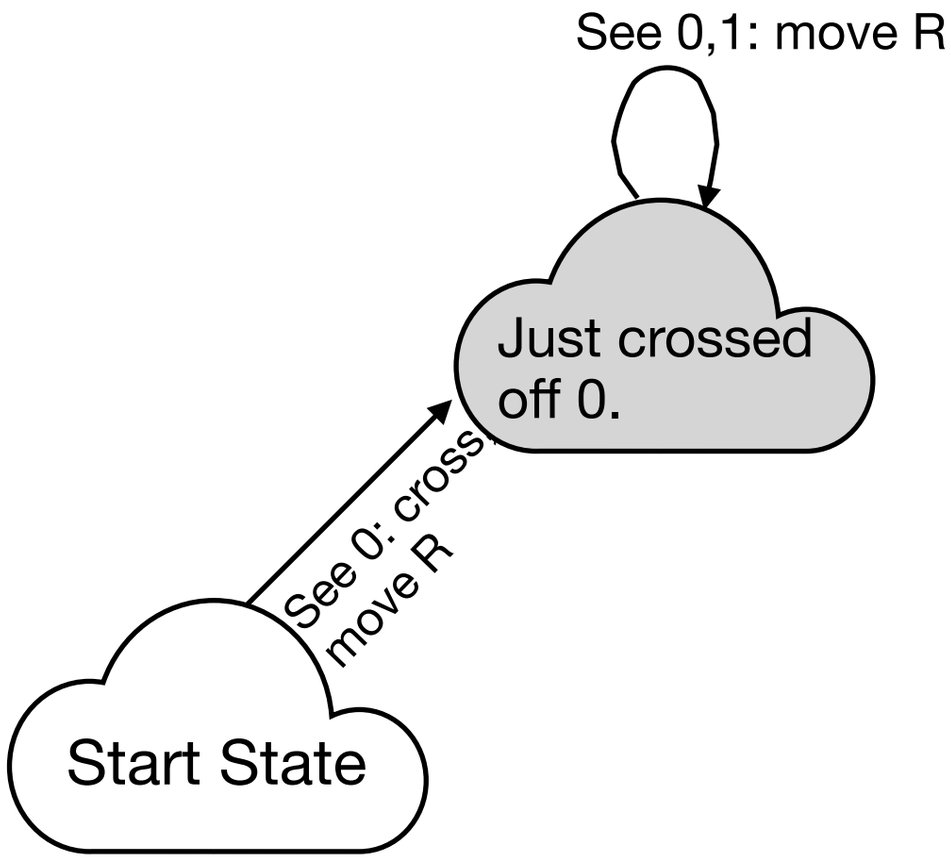

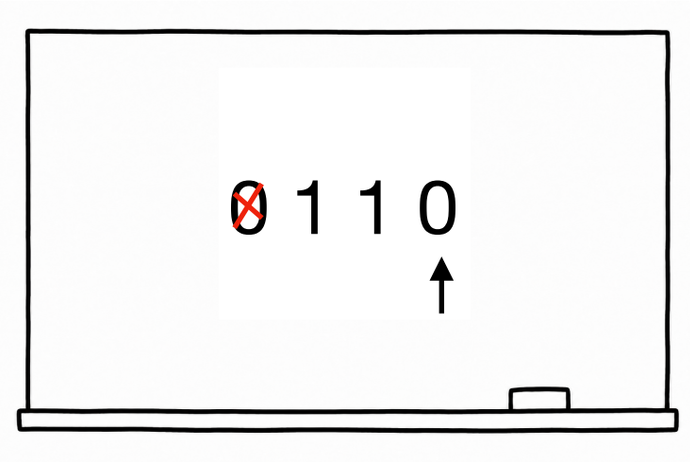

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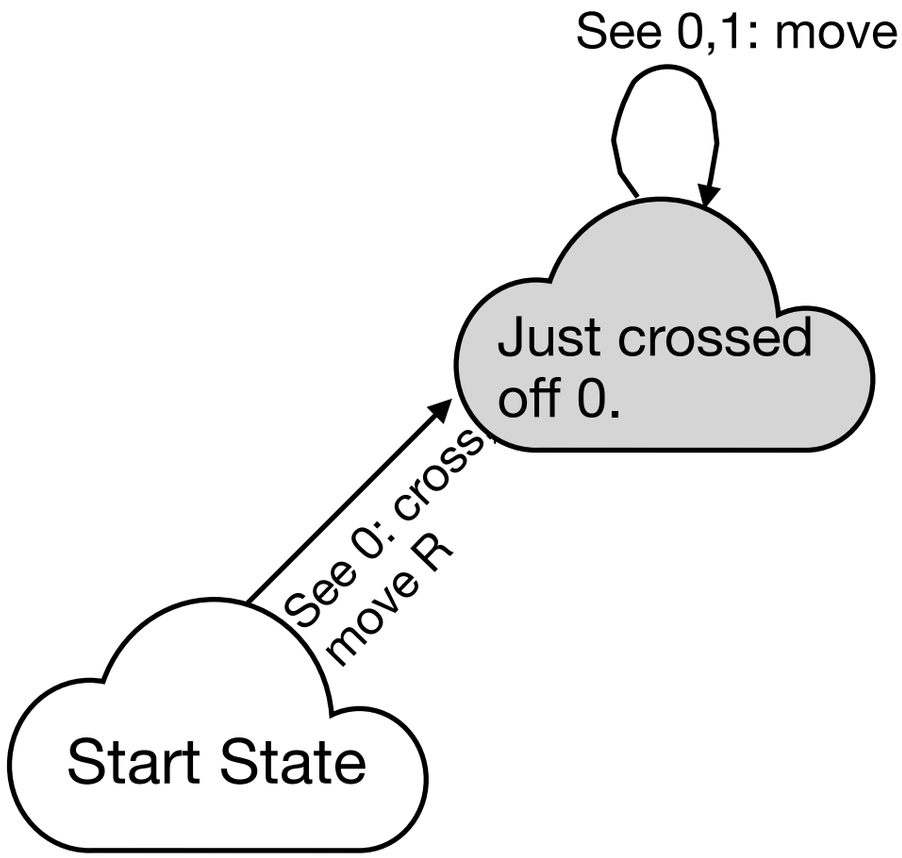

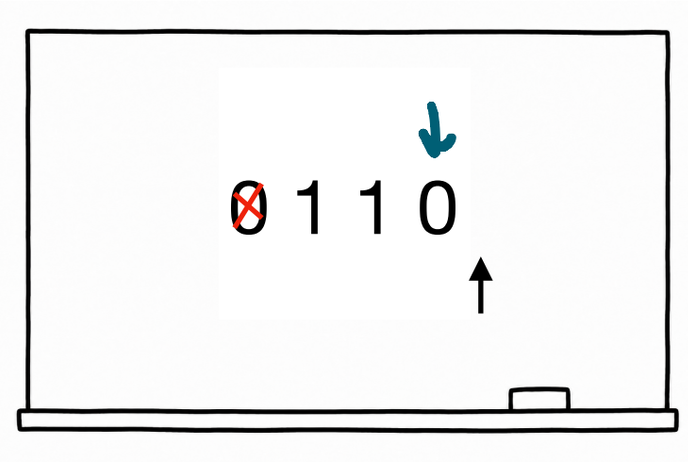

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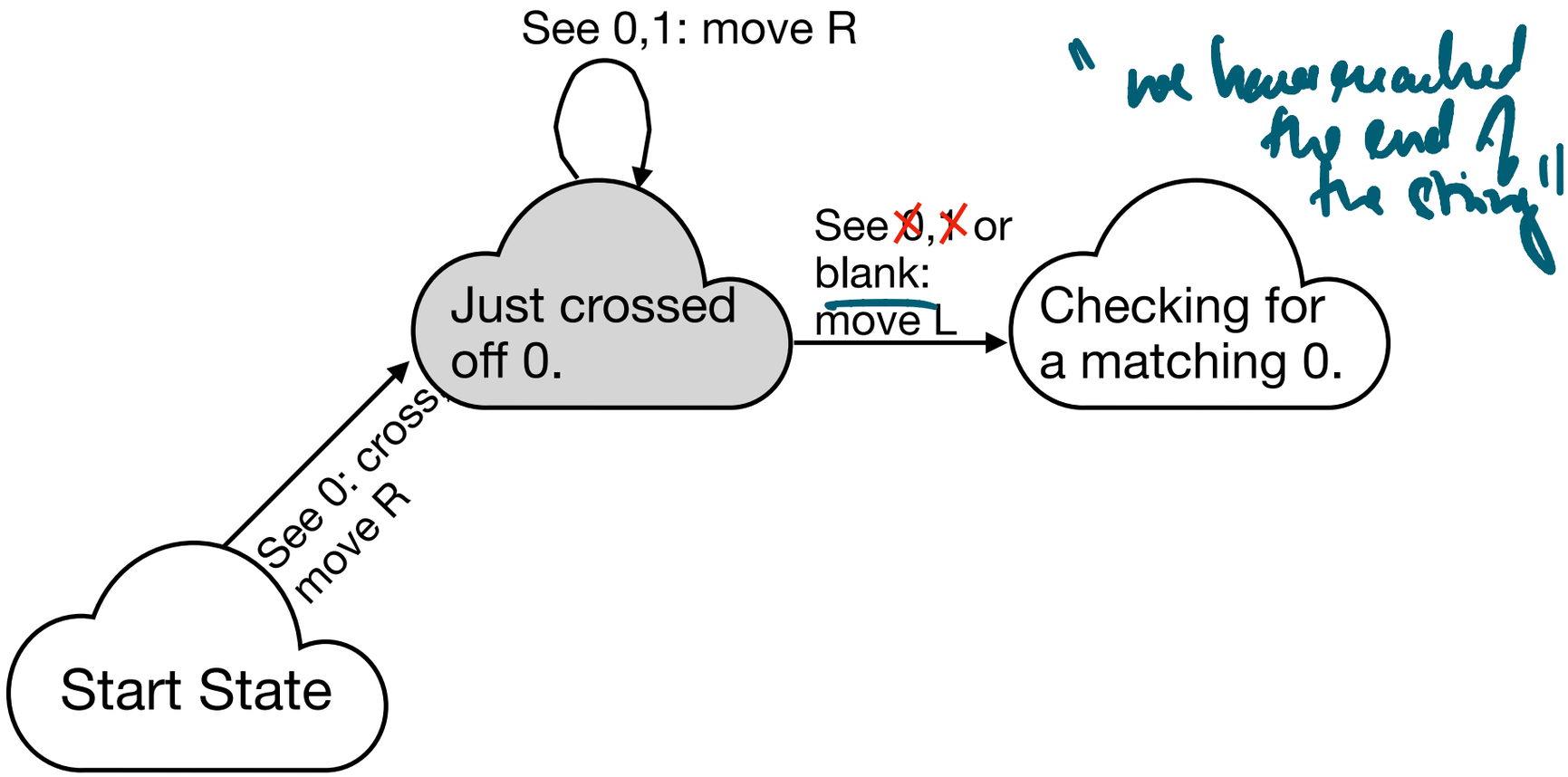
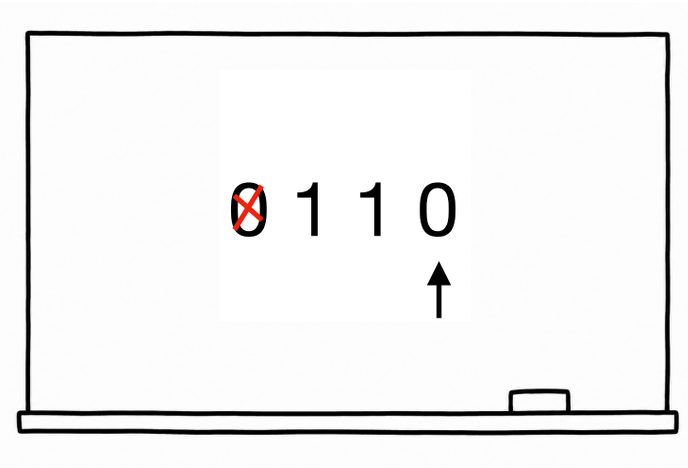


" we have reached the end of the string"



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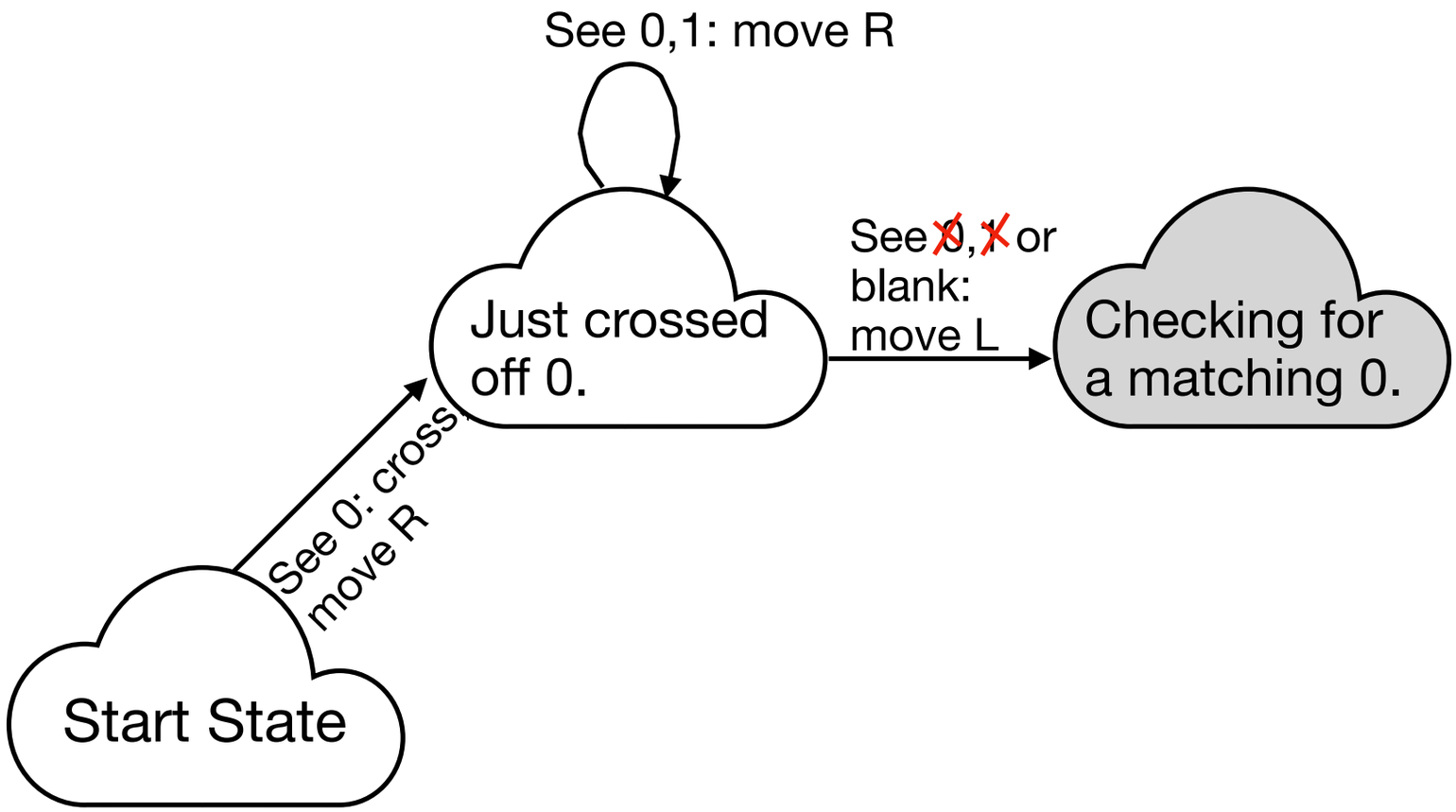

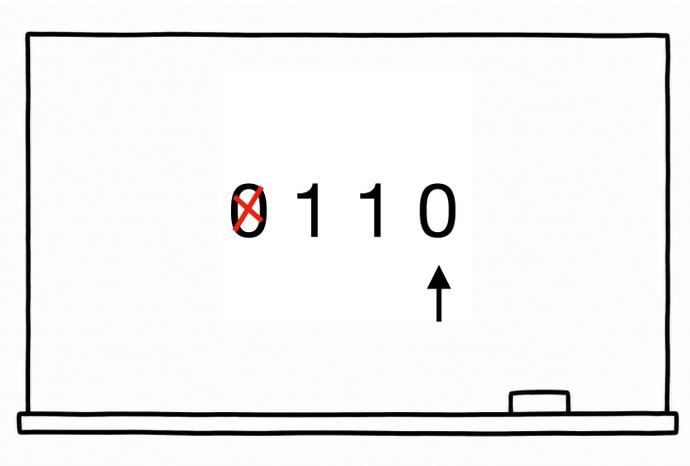

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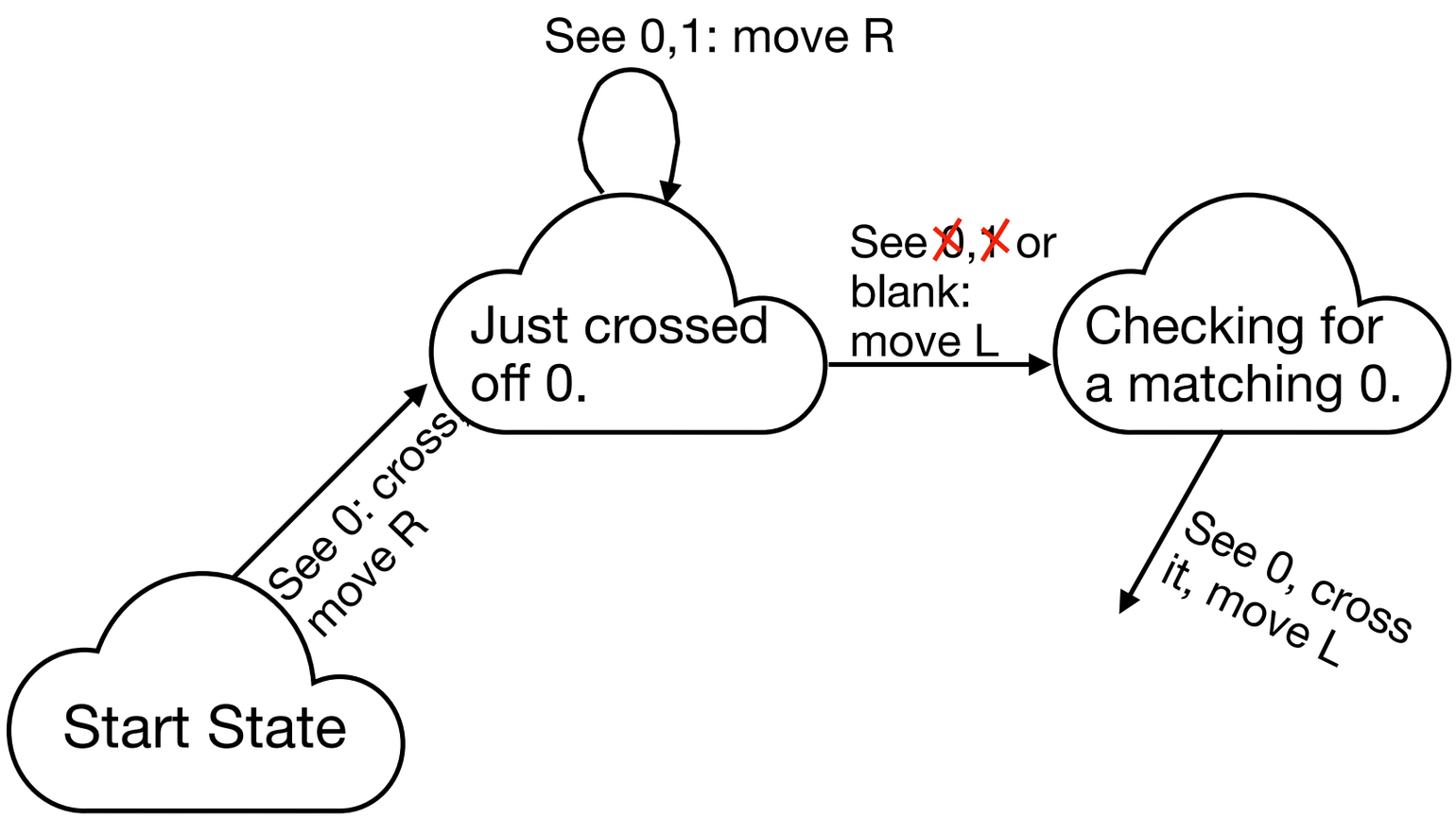

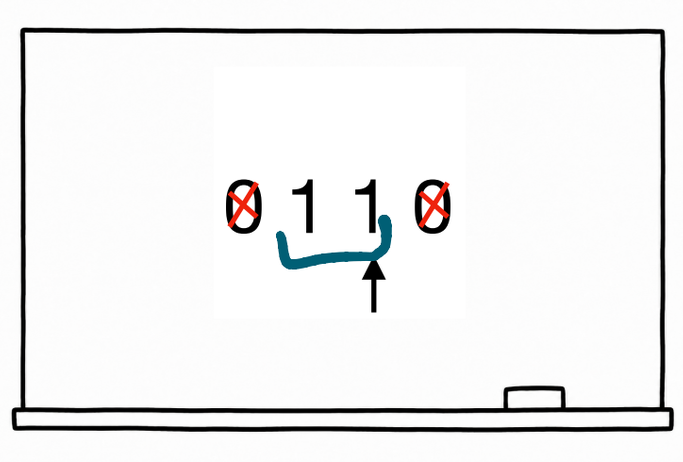

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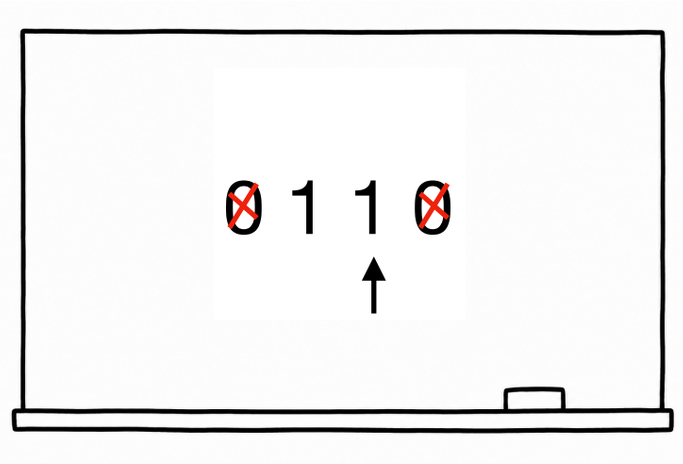

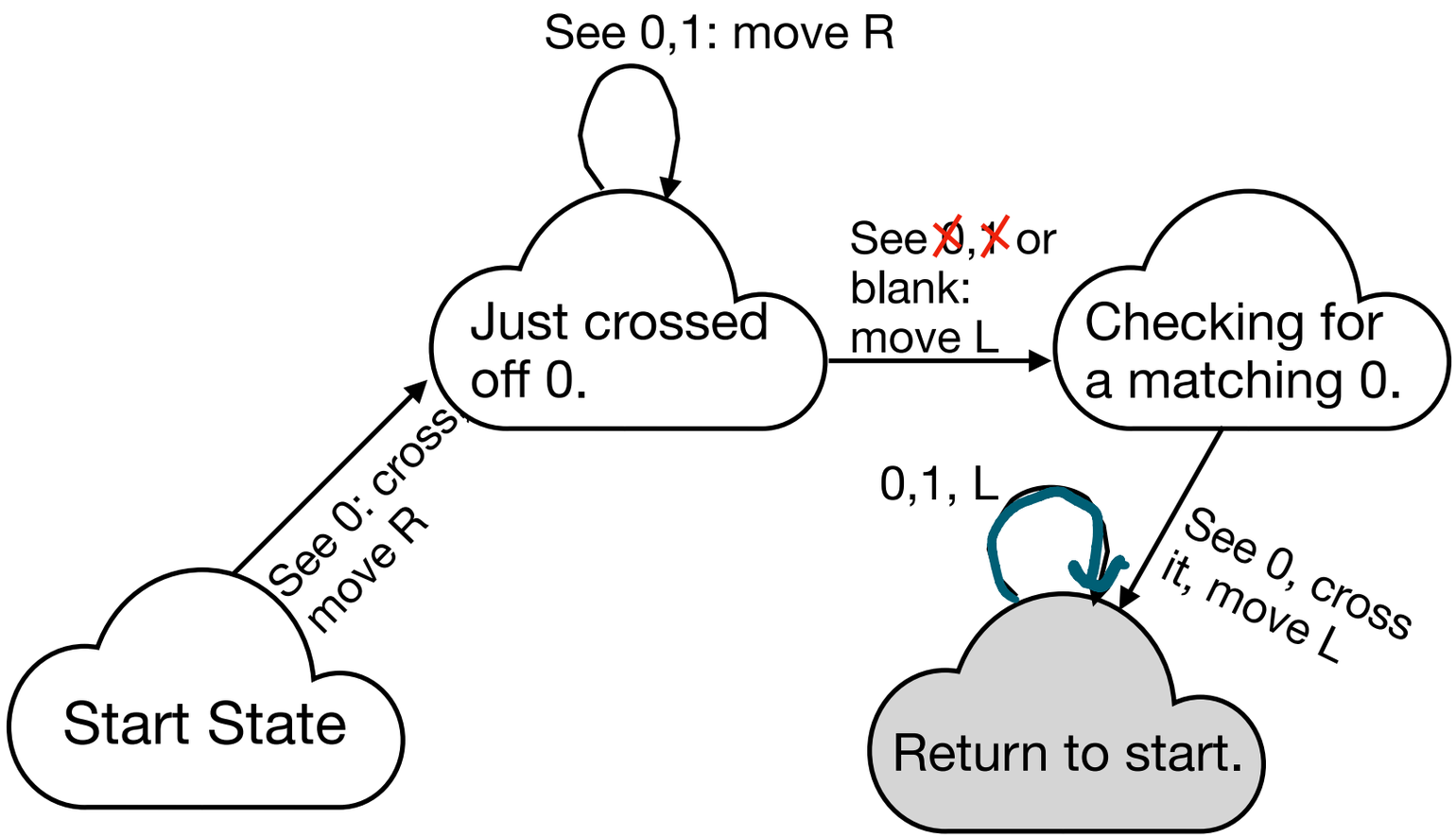

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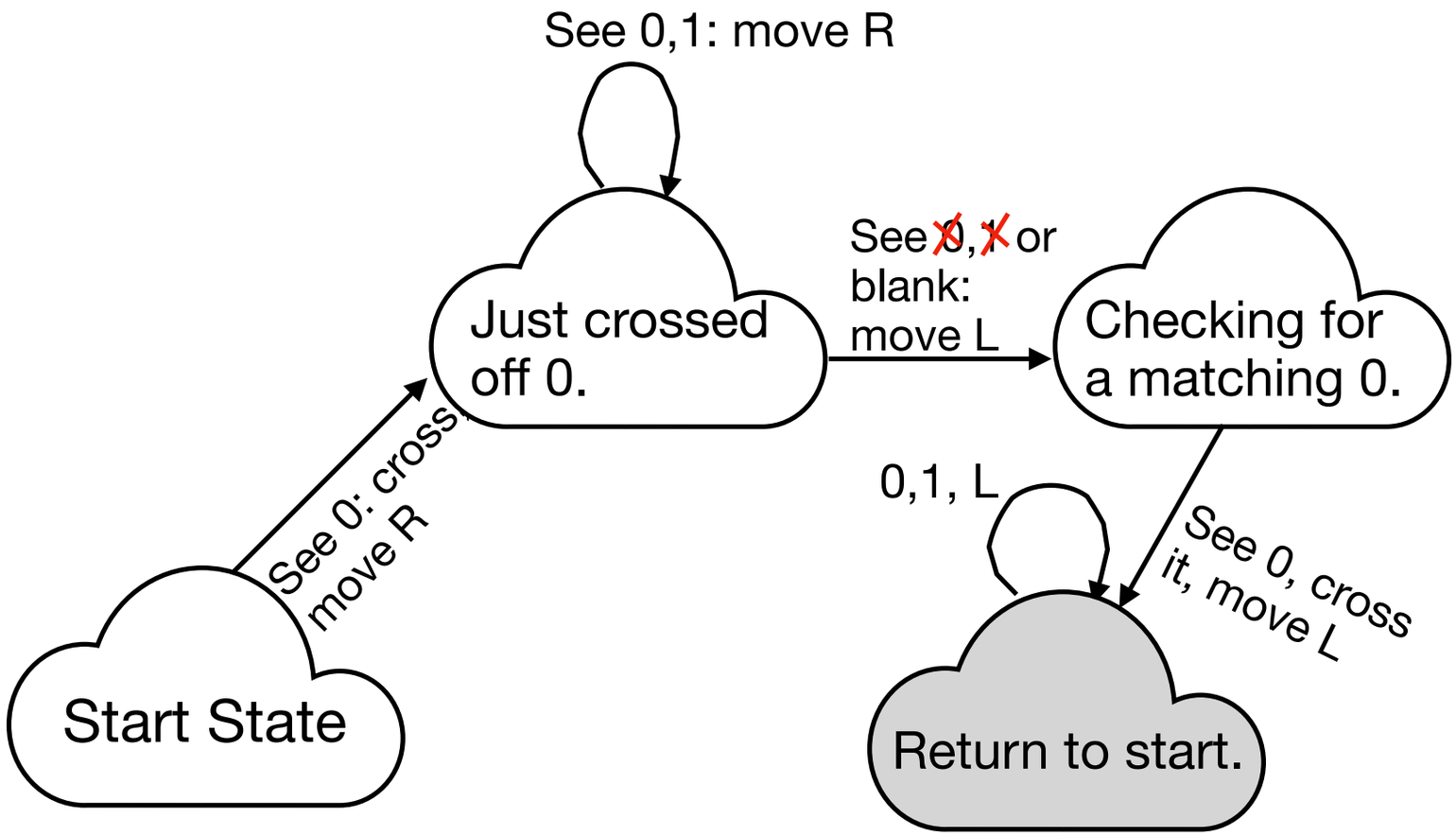

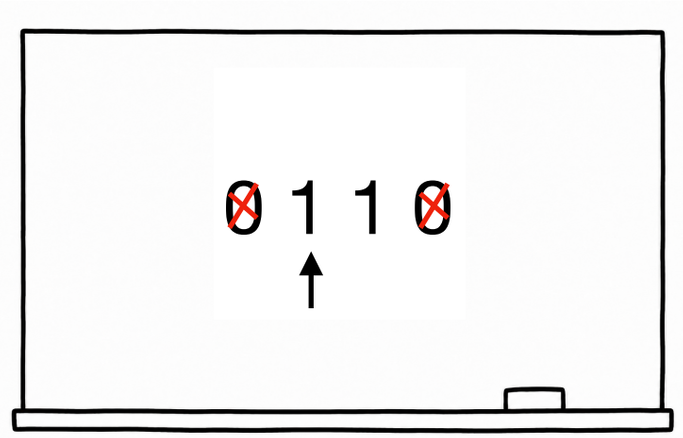

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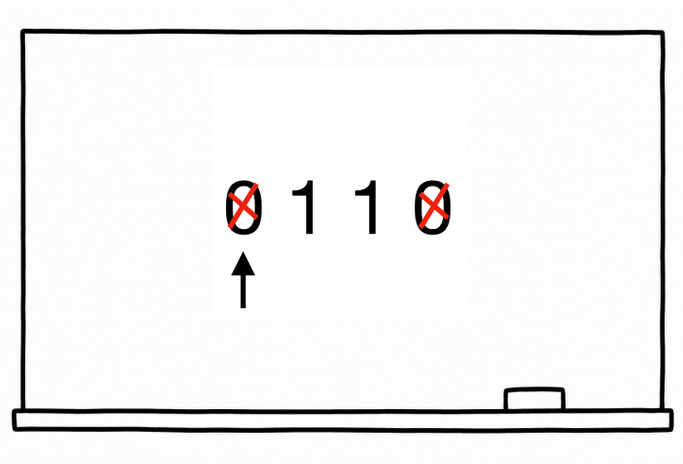

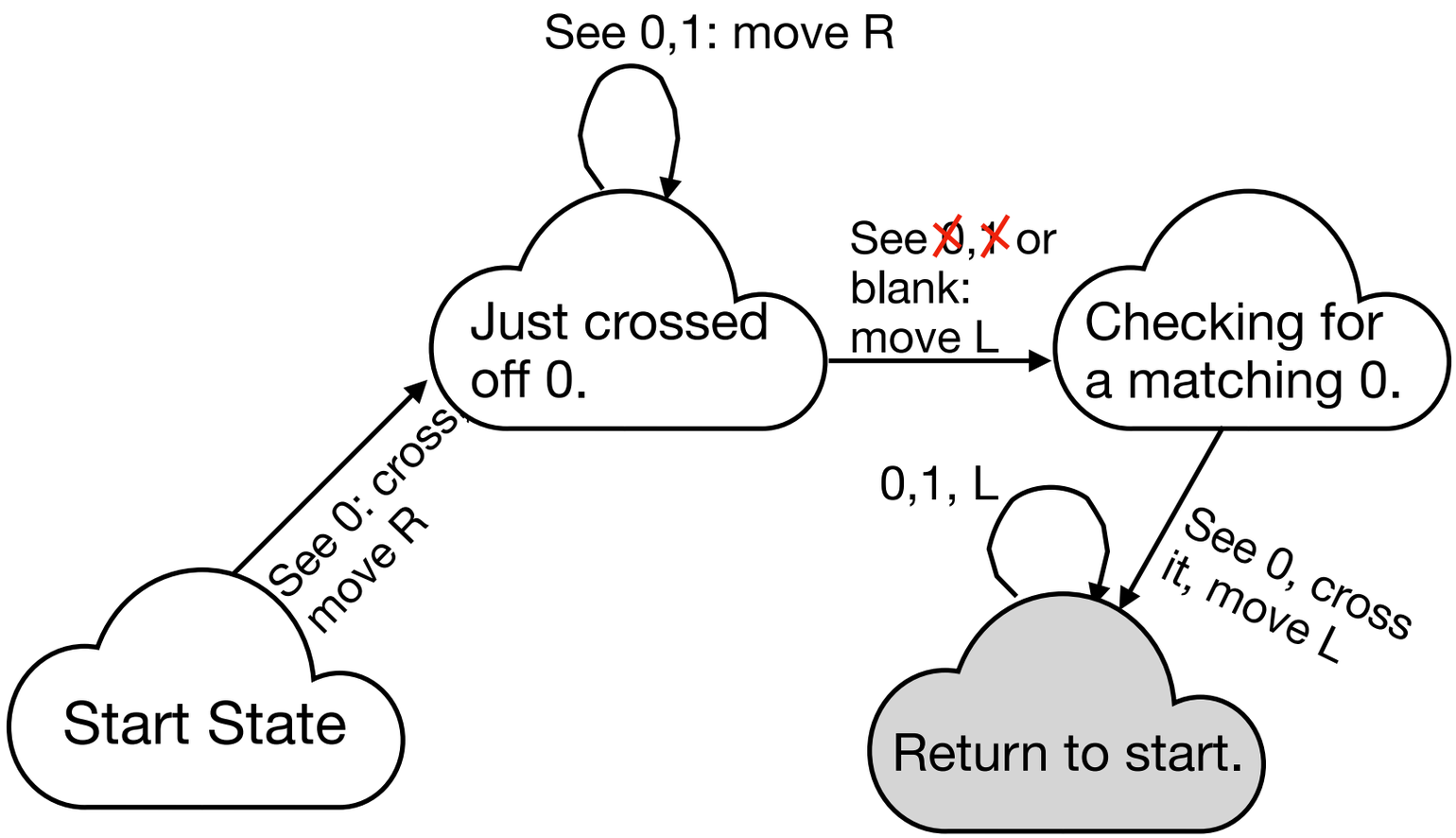

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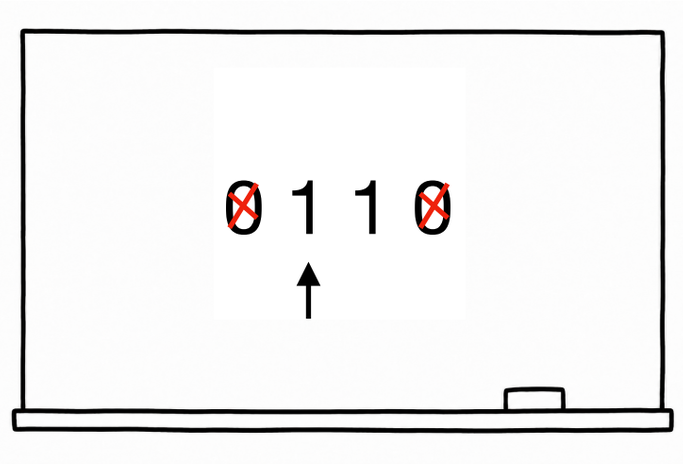

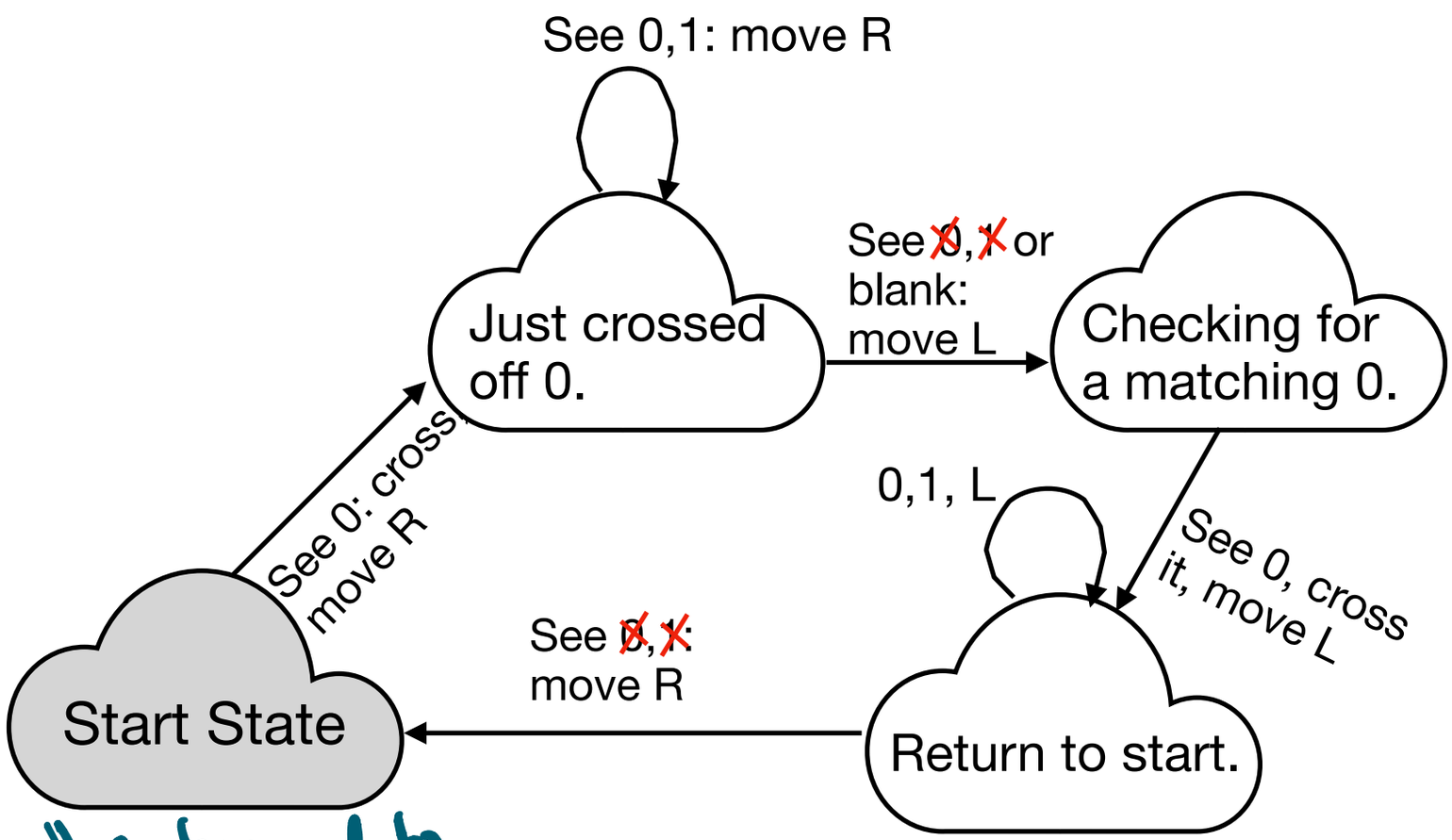

" returned to the beginning of the string".

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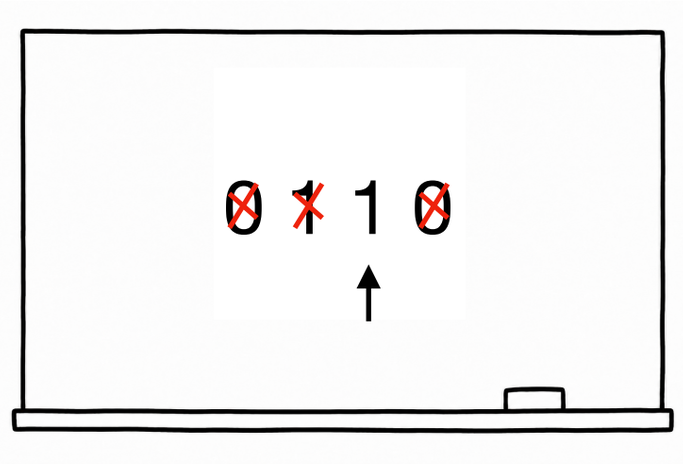

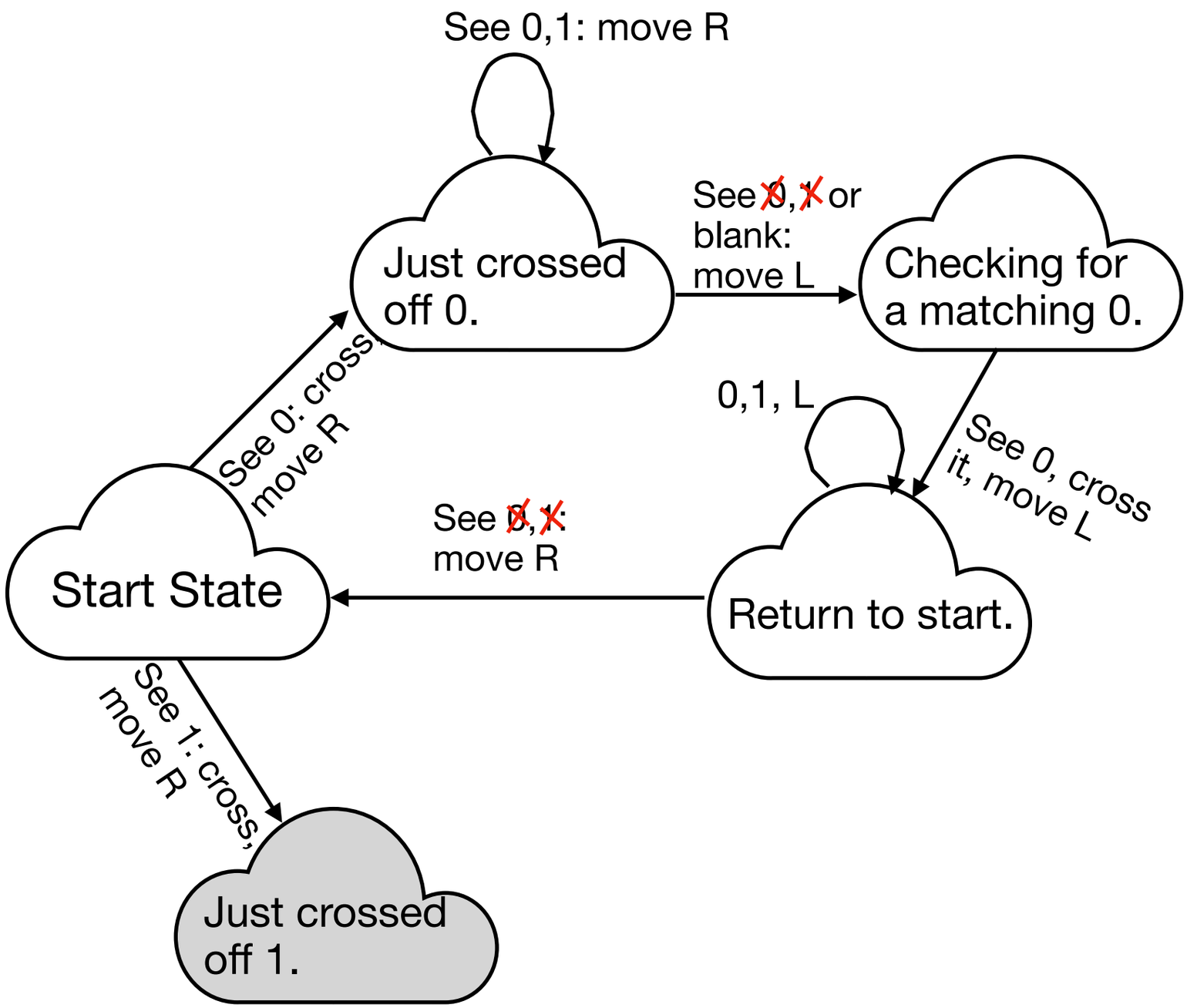

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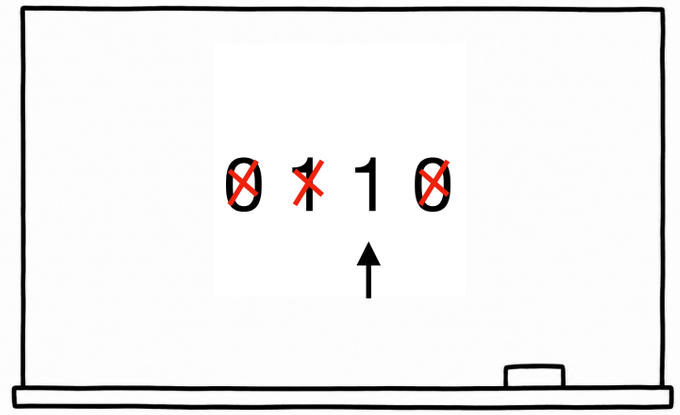

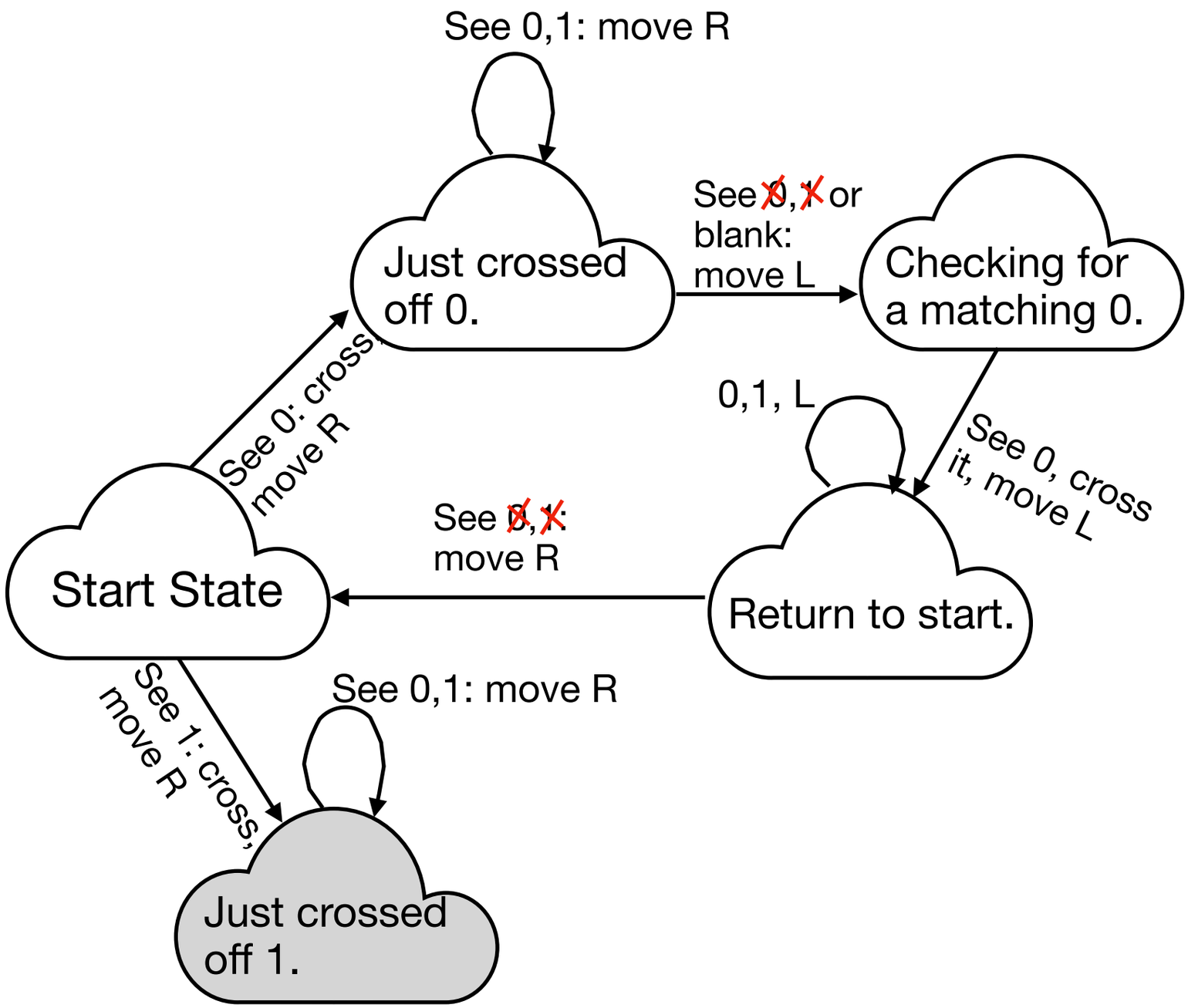

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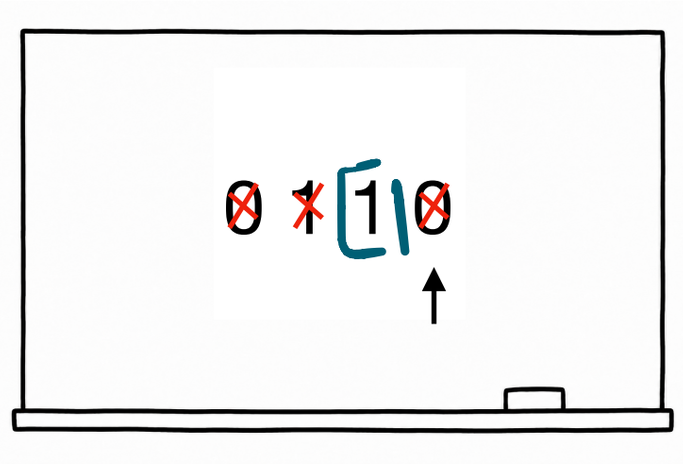

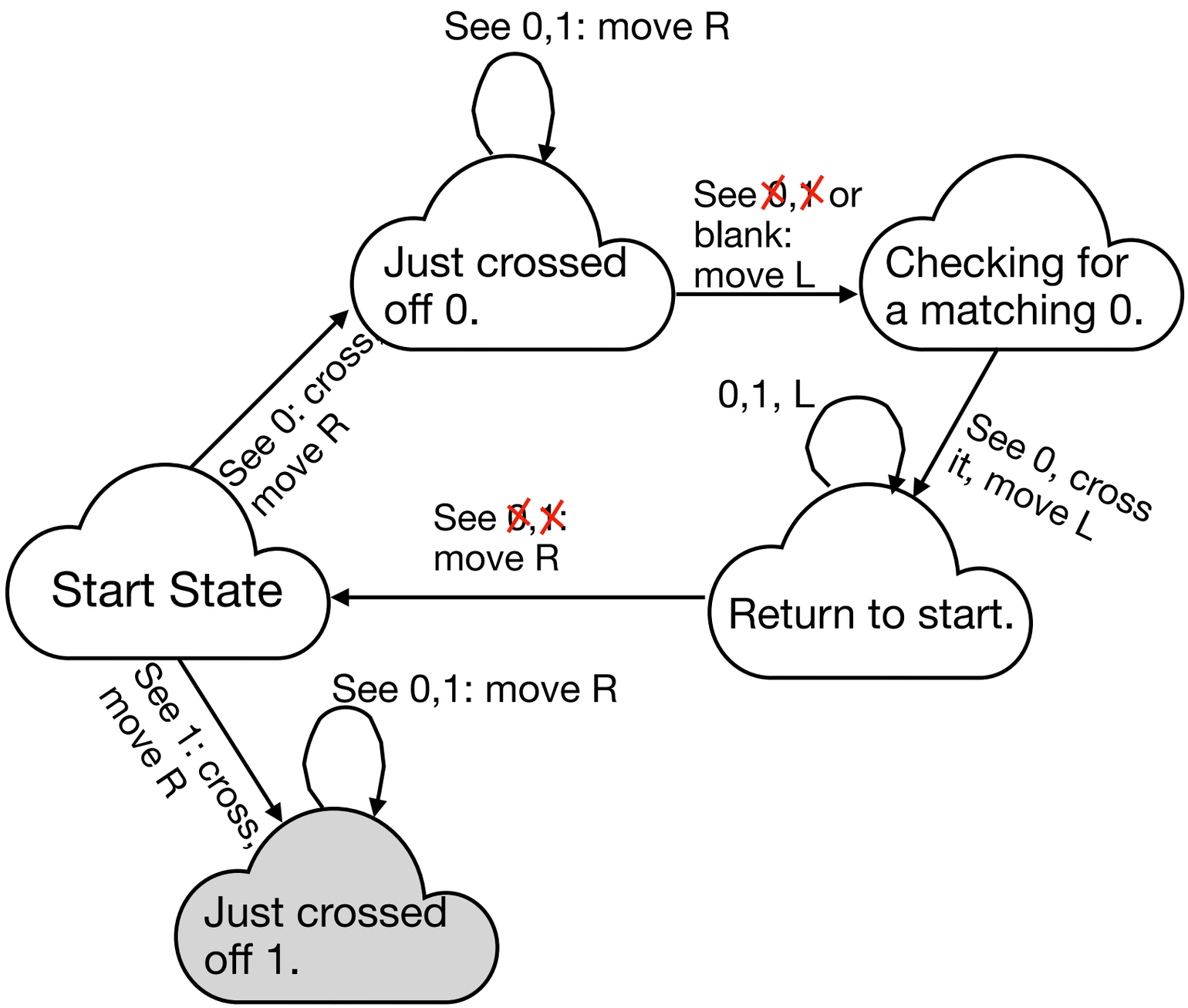

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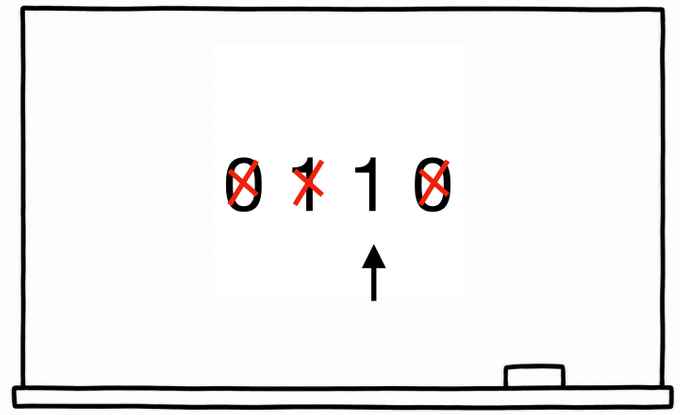

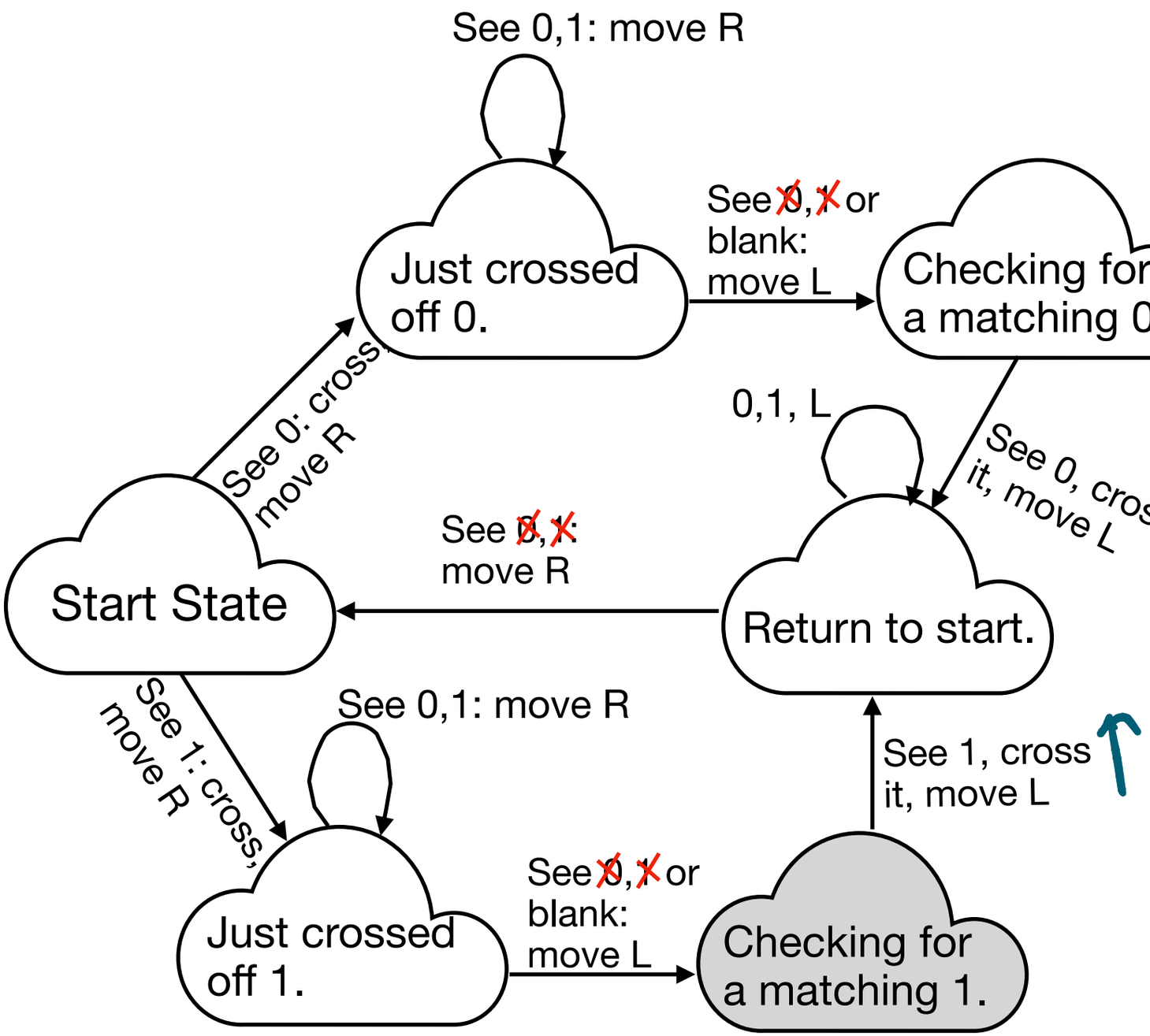

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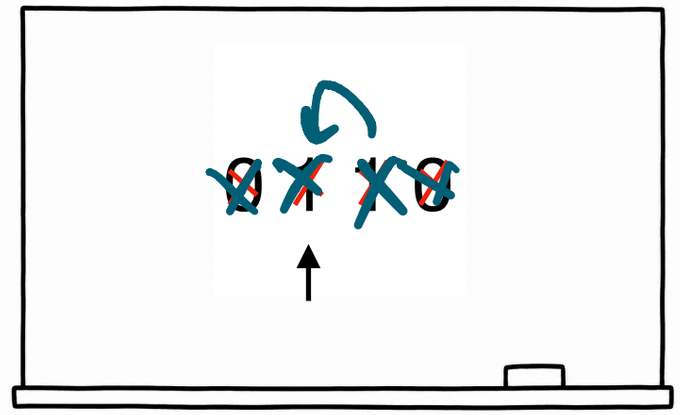

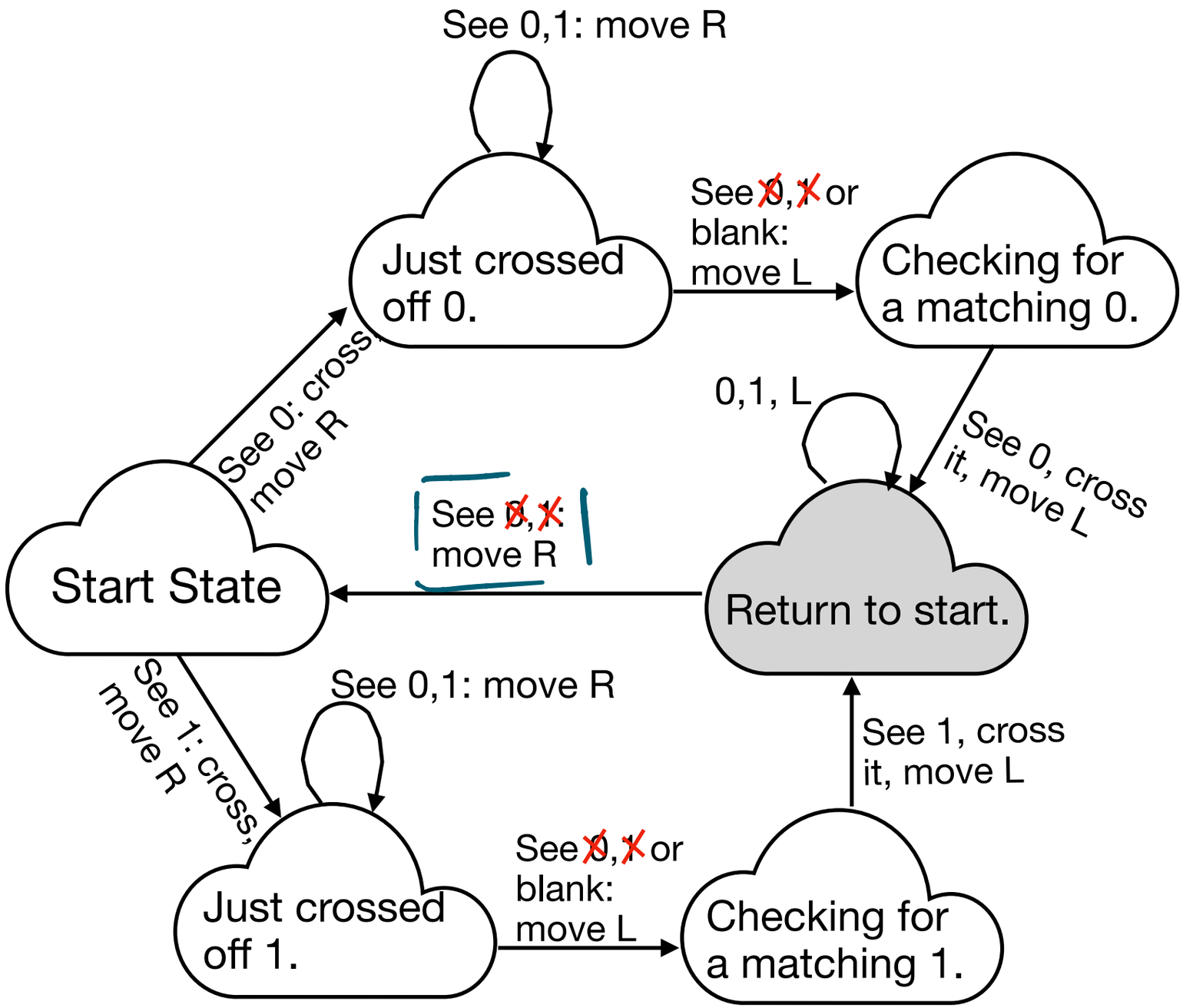

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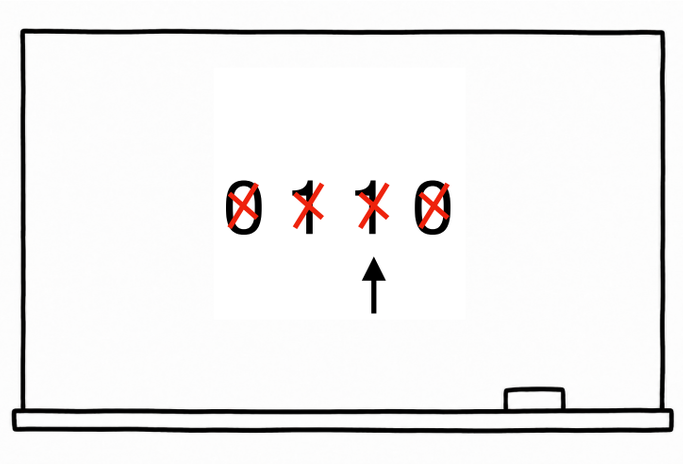
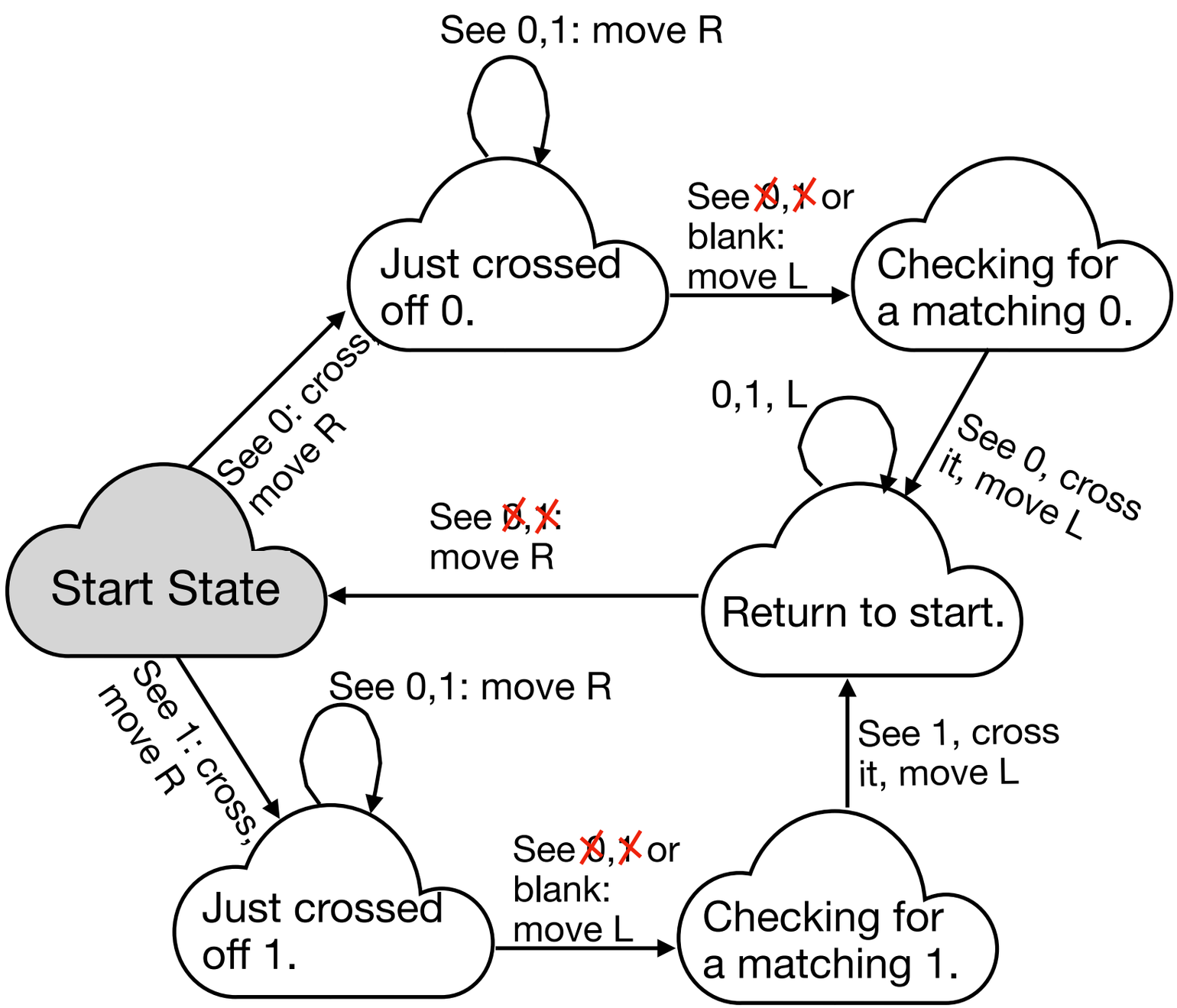
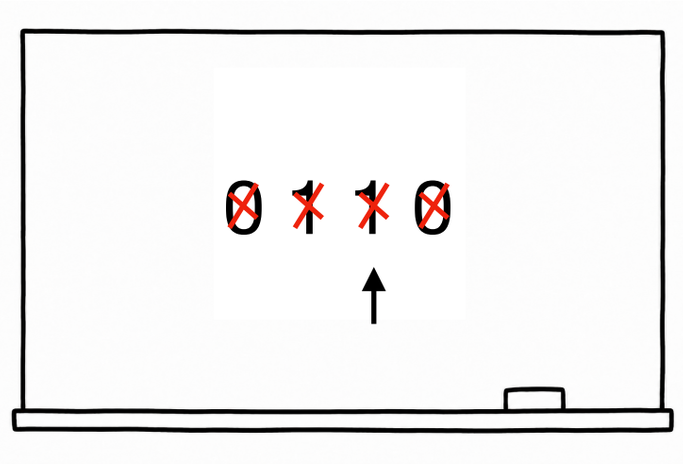
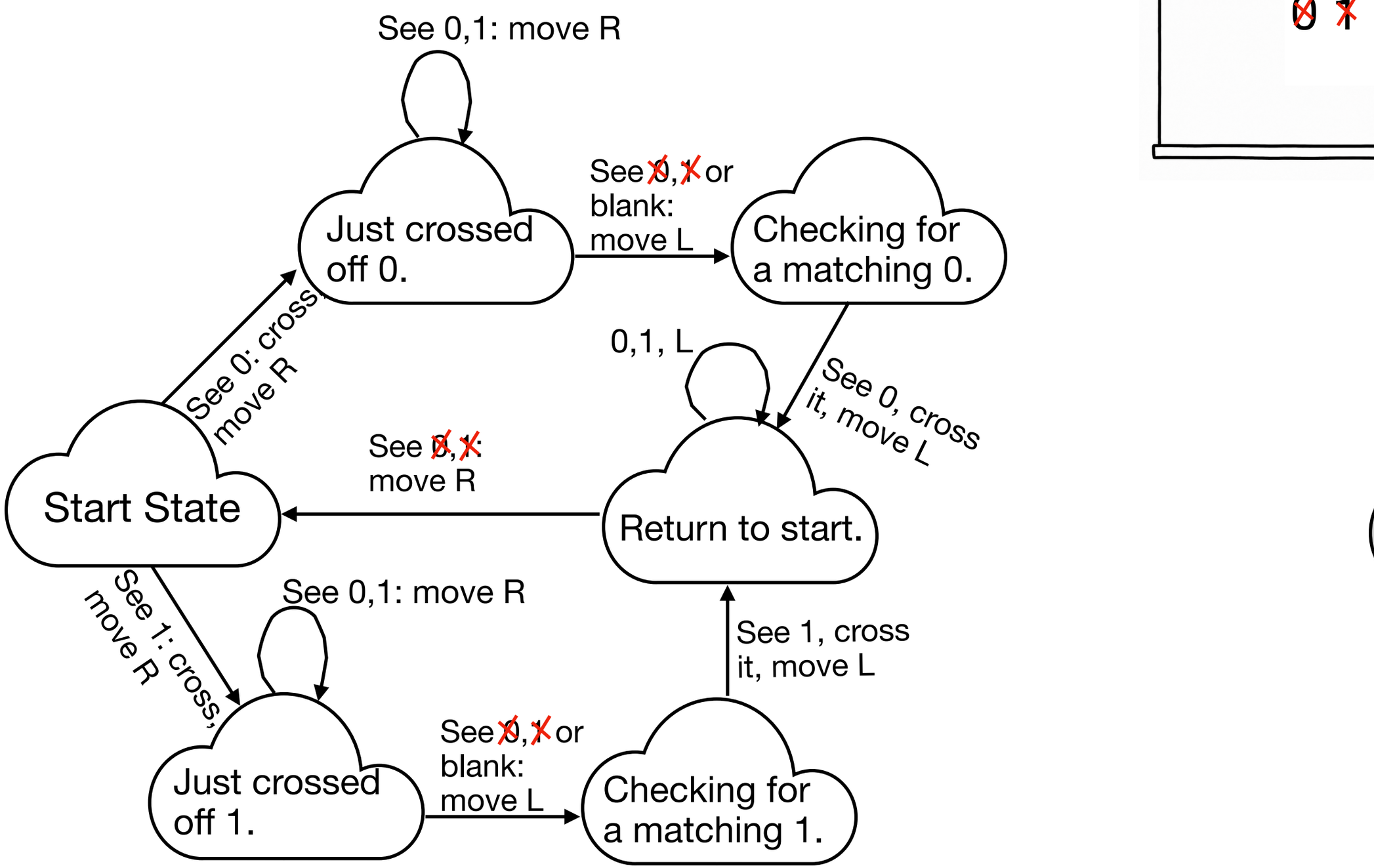
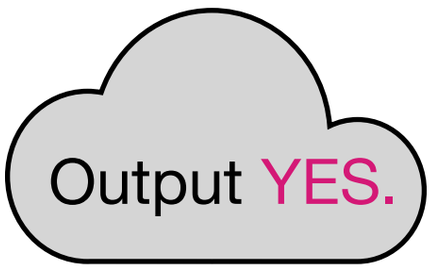




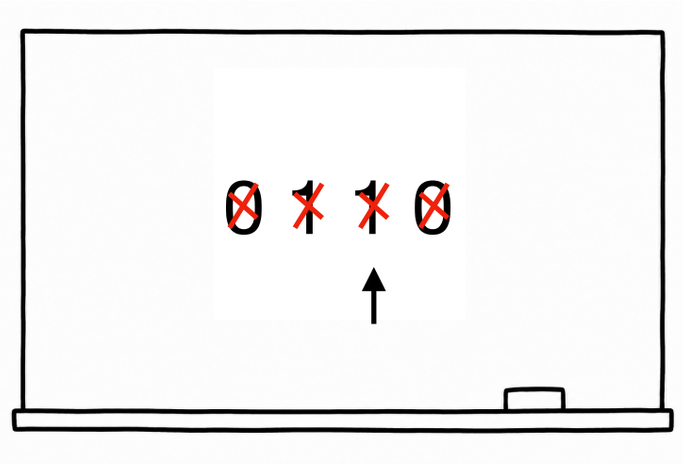
State Diagram



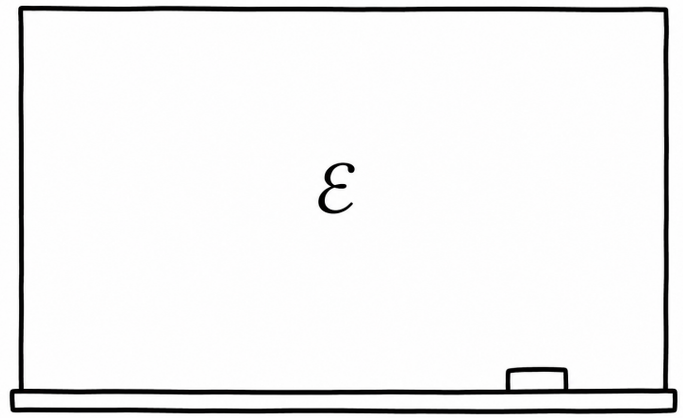




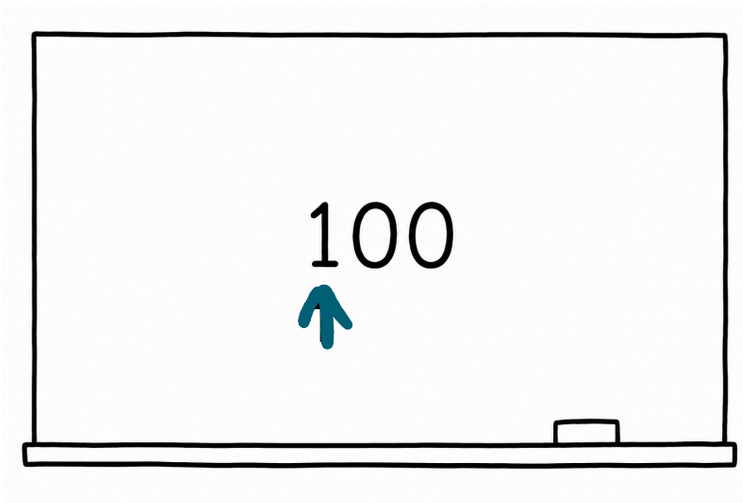
State Diagram



State Diagram



Examples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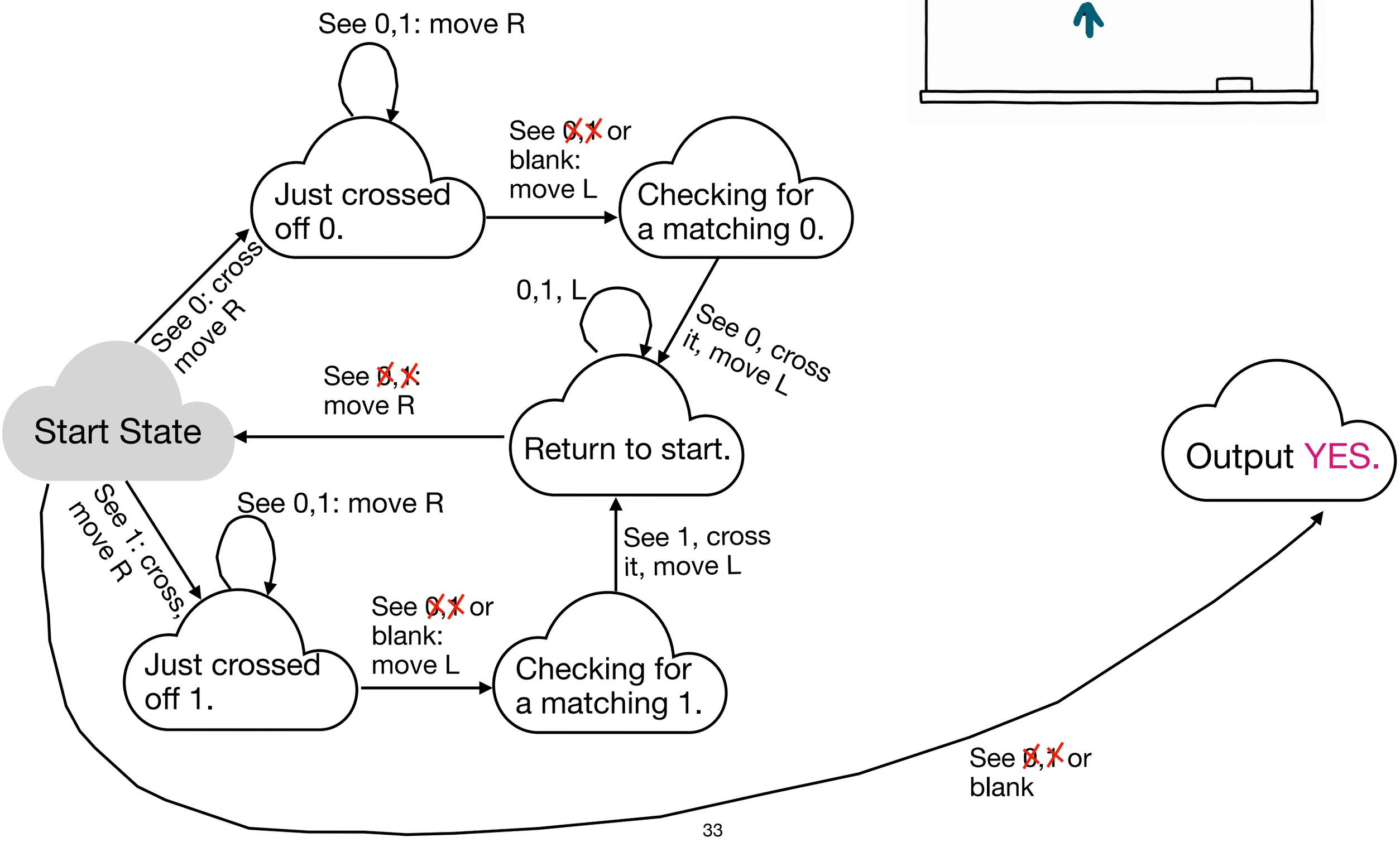
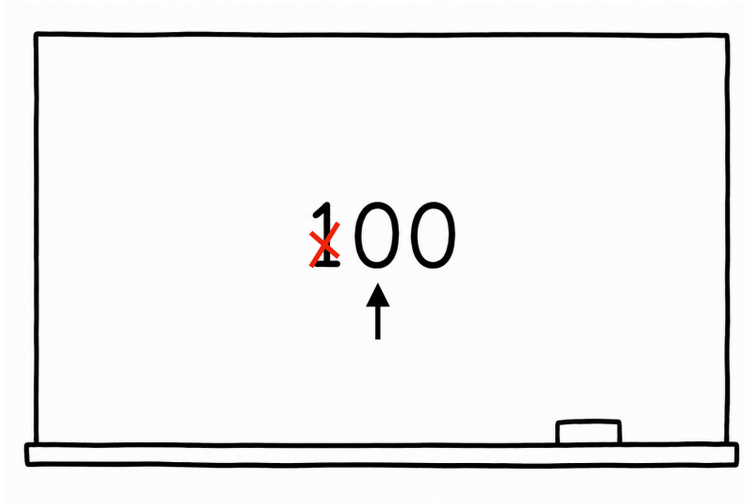

See 0,1: move R
Just crossed off 0.
See 0,1 or blank: move L
Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
See 0,1: move R
Start State
Return to start.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
Just crossed off 1.
See 0,1 or blank: move L
Checking for a matching 1.
Output YES.
See 0,1 or blank

Examples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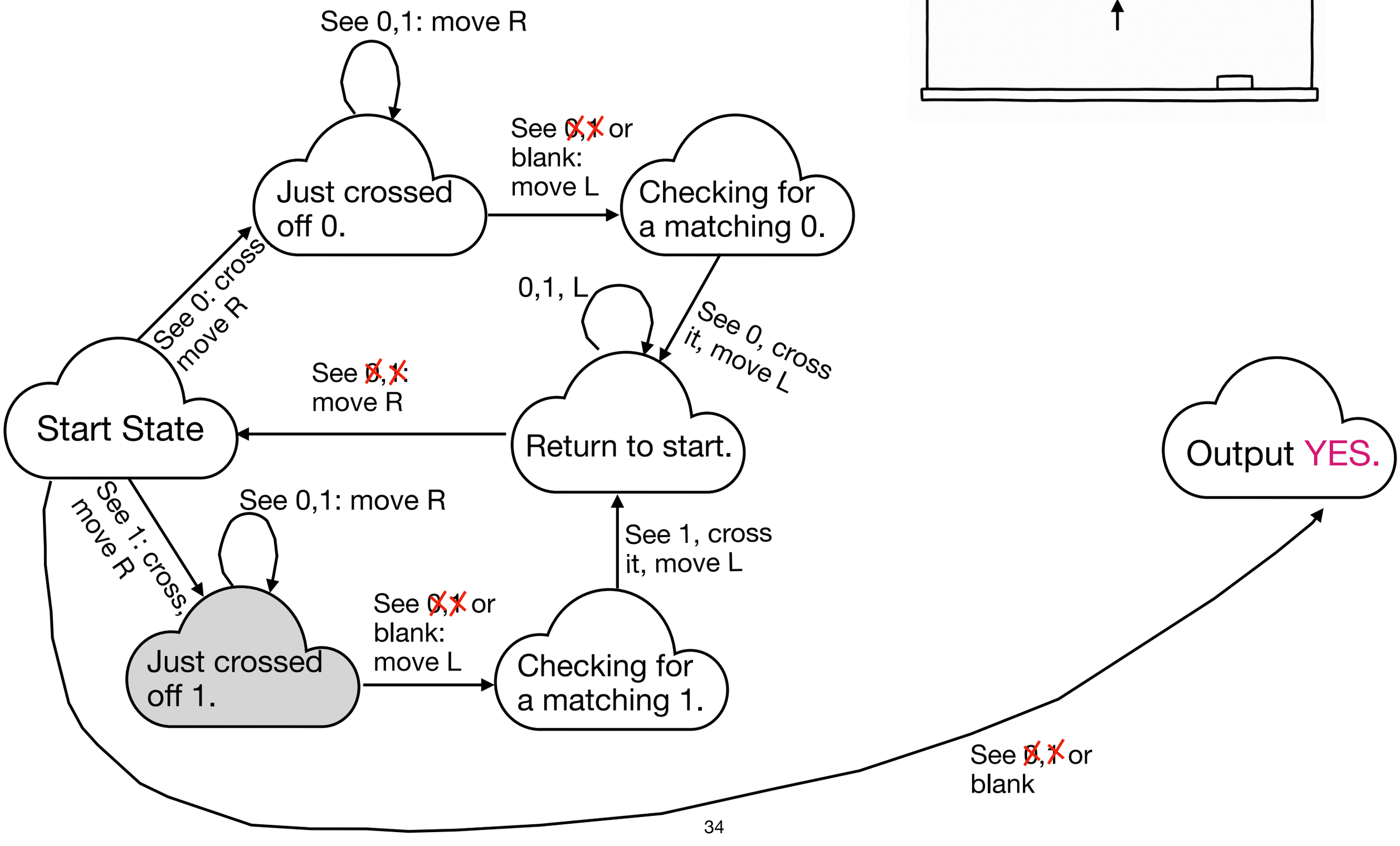
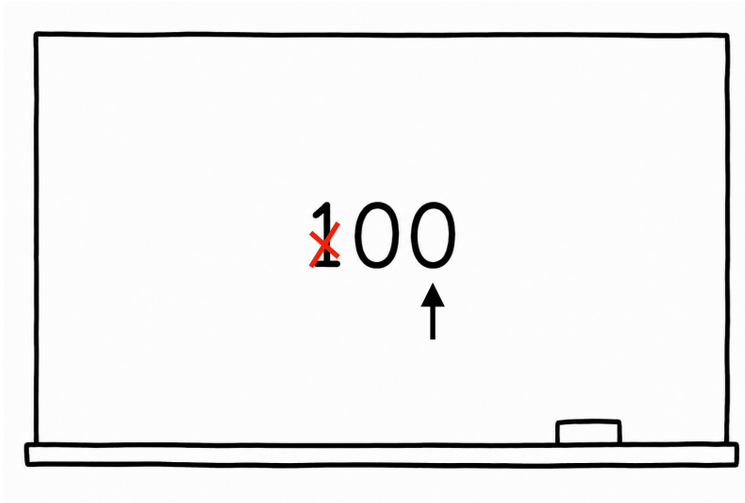

See 0,1: move R
Just crossed off 0.
See 0, 1 or blank: move L
Checking for a matching 0.
See 0: cross, move R
0,1, L
See 0, cross it, move L
See 0, 1: move R
Start State
Return to start.
Output YES.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
Just crossed off 1.
See 0, 1 or blank: move L
Checking for a matching 1.
See 0, 1 or blank

Examples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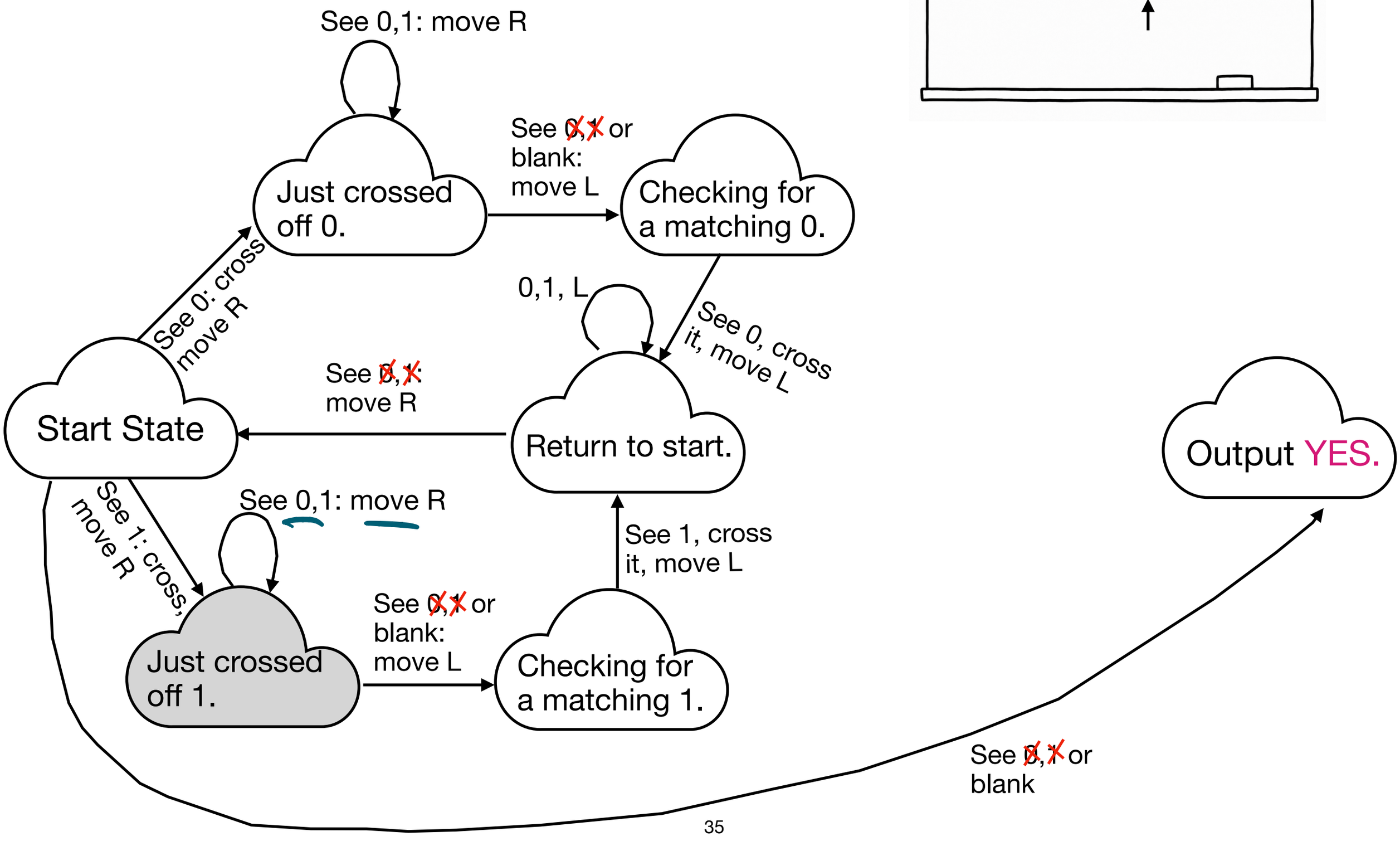
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Just crossed off 0.
See 0,1 or blank: move L
Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
See 0,1: move R
Start State
Return to start.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
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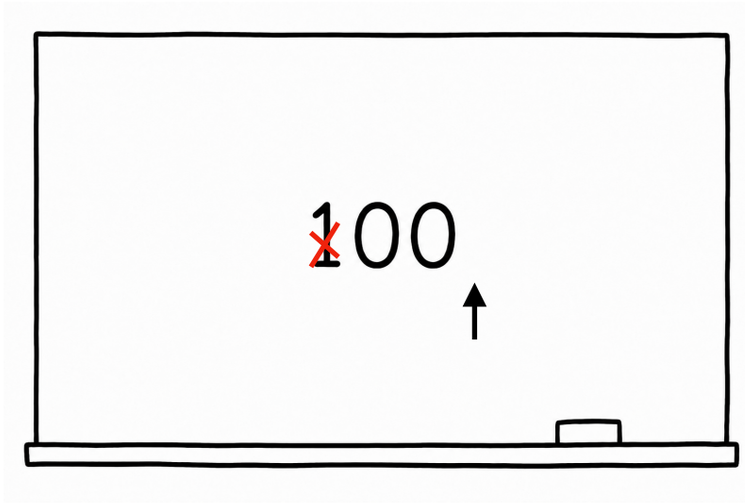
See 0,1 or blank: move L
Checking for a matching 1.
Output YES.
See 0,1 or blank

Examples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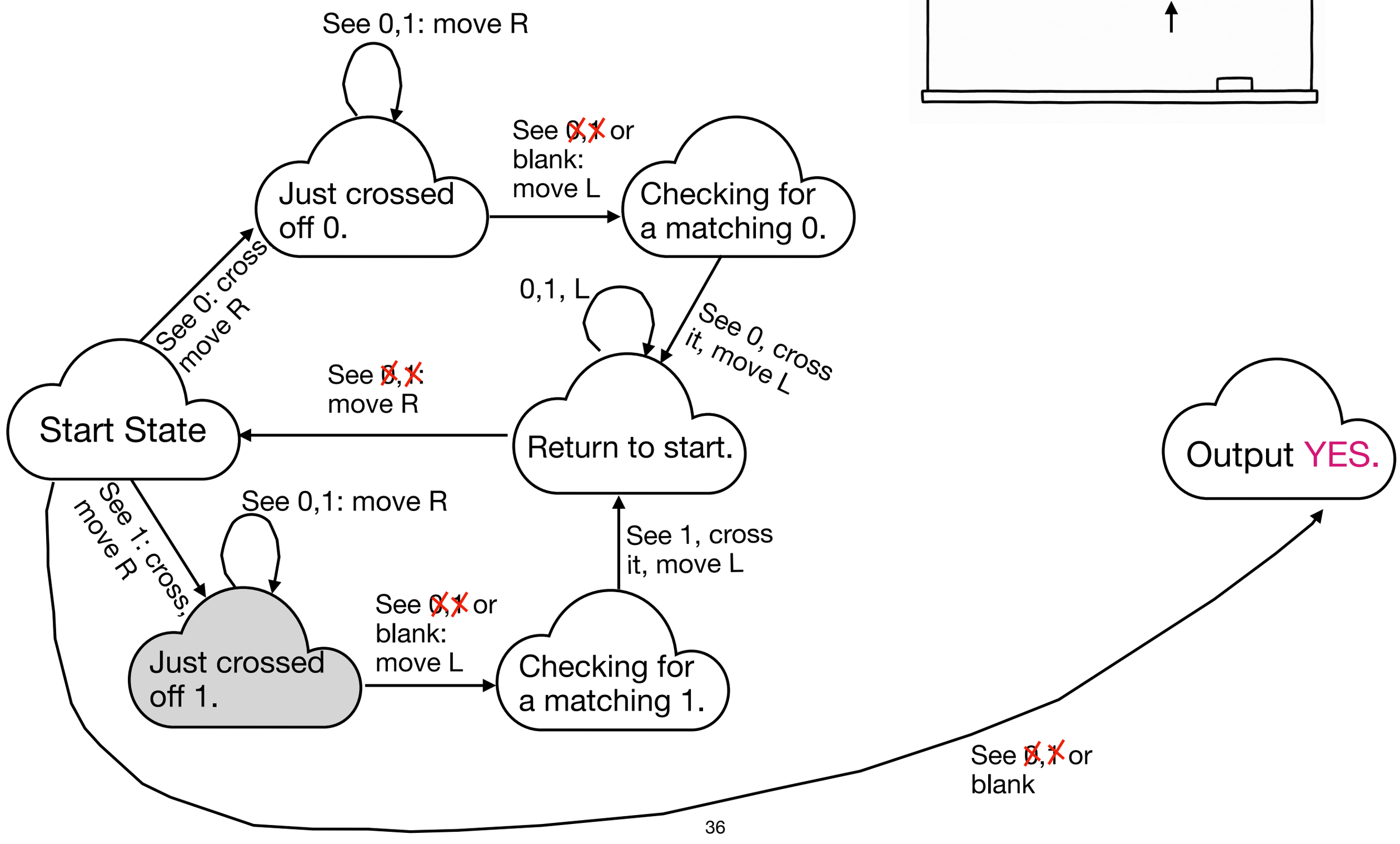
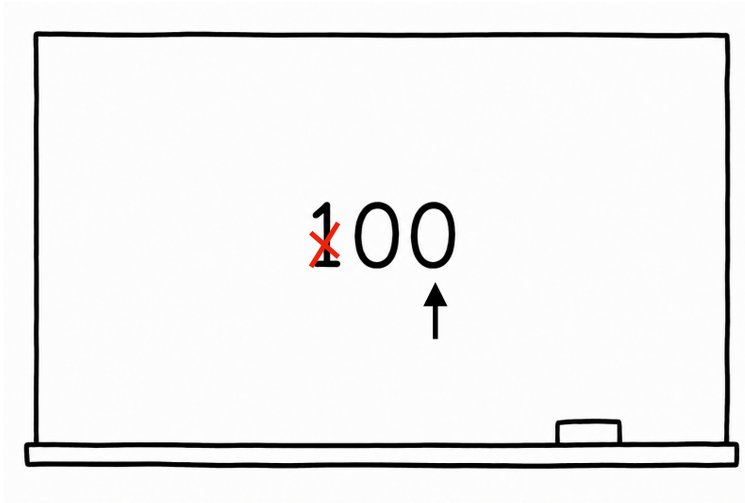

See 0,1: move R
Just crossed off 0.
See 0, 1 or blank: move L
Checking for a matching 0.
0,1, L
See 0, cross it, move L
See 0: cross, move R
See 0, 1: move R
Start State
Return to start.
Output YES.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
Just crossed off 1.
See 0, 1 or blank: move L
Checking for a matching 1.
See 0, 1 or blank

Examples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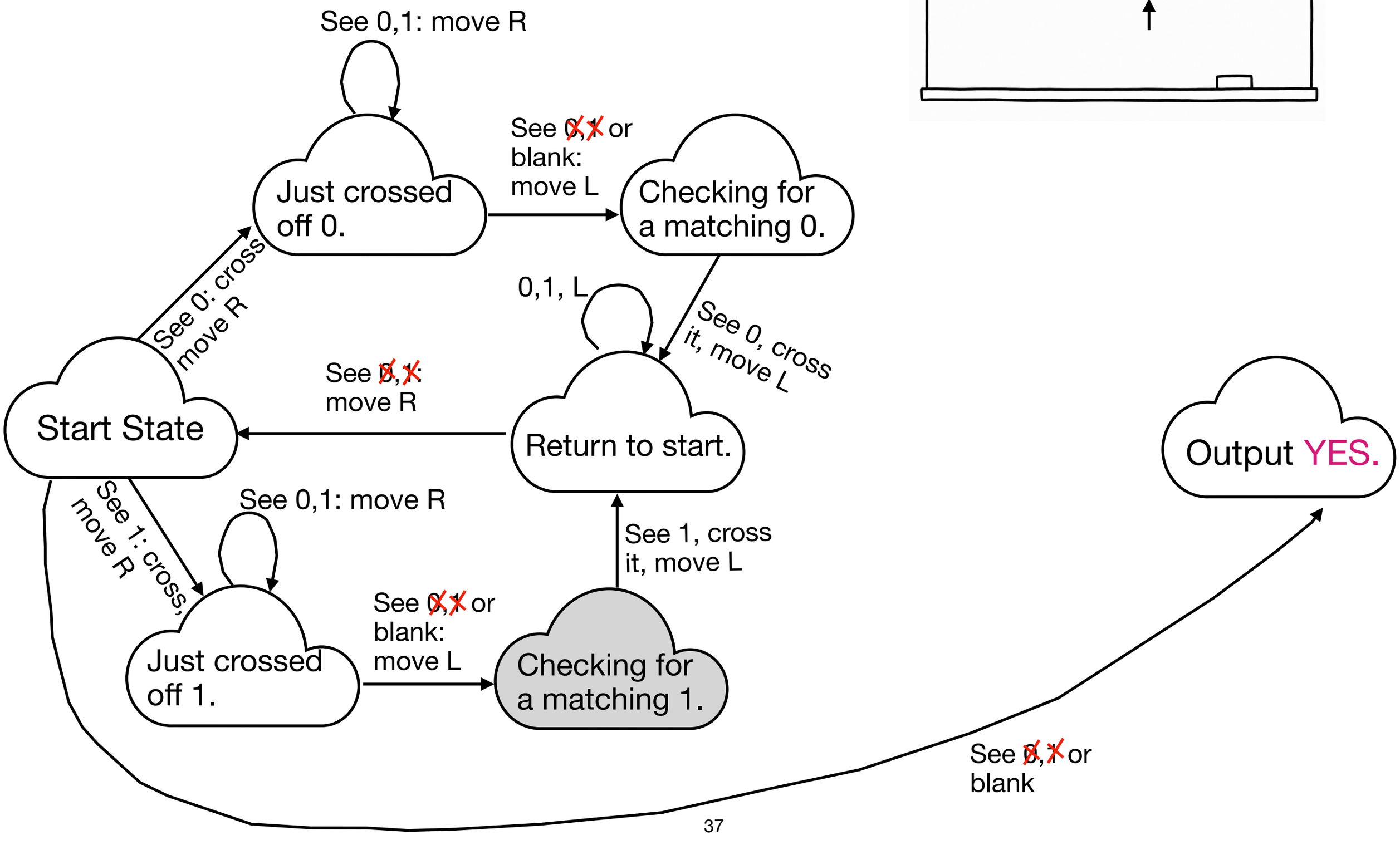
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See 0,1 or blank: move L
Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
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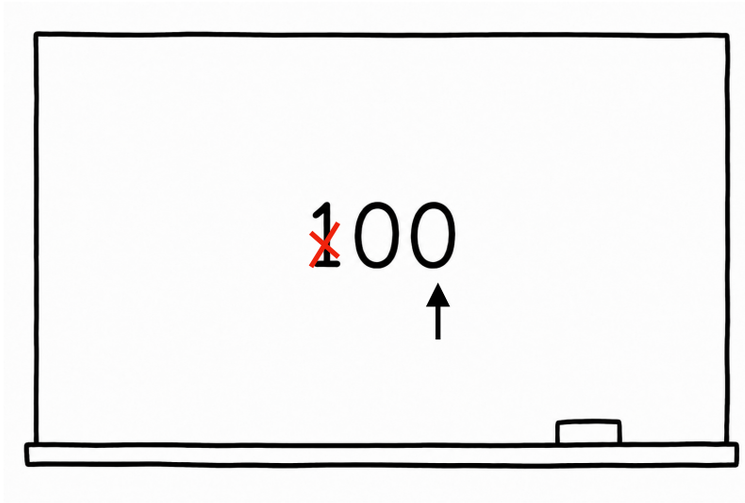
Start State
Return to start.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
Just crossed off 1.
See 0,1 or blank: move L
Checking for a matching 1.
Output YES.
See 0,1 or blank

Examples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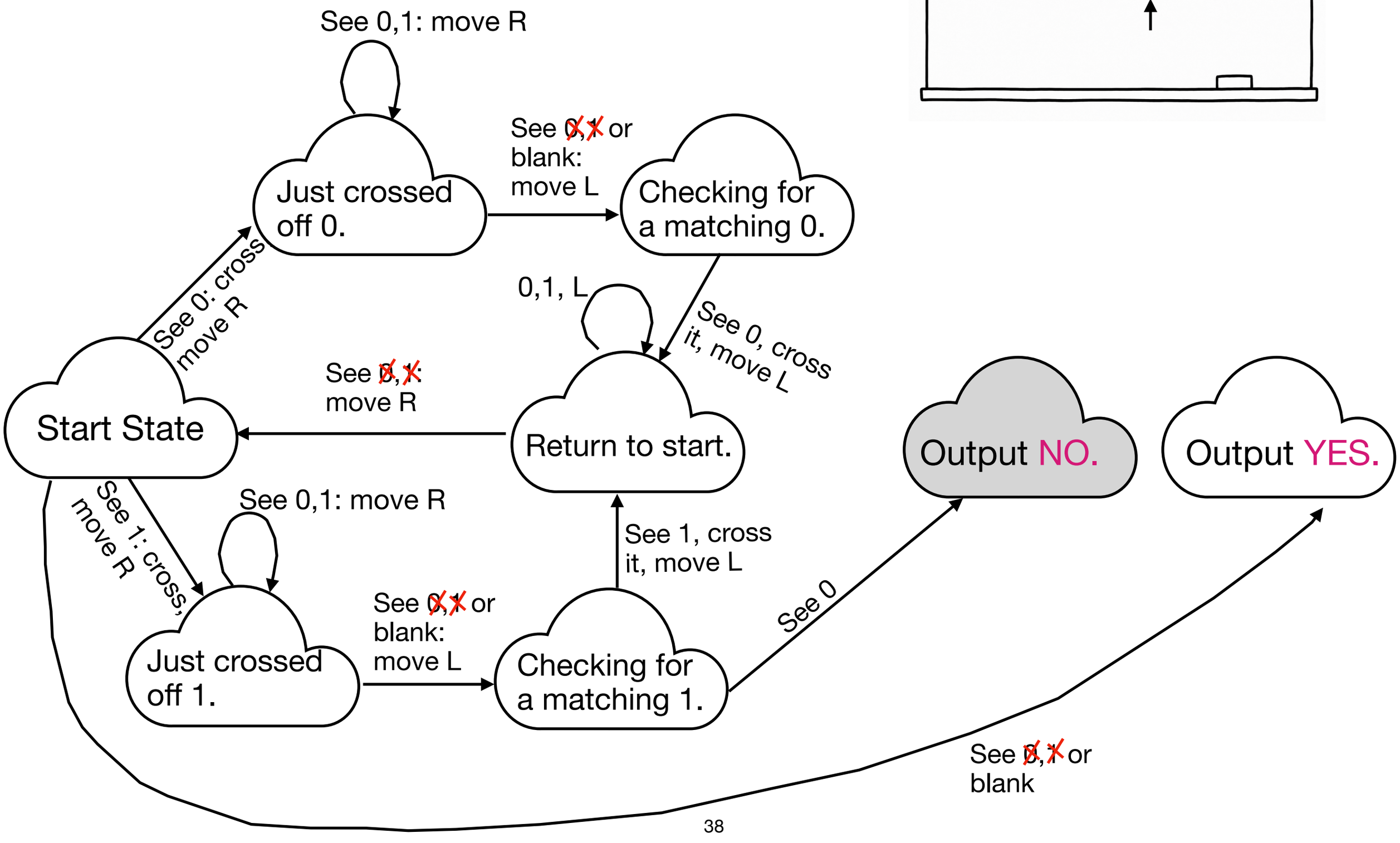
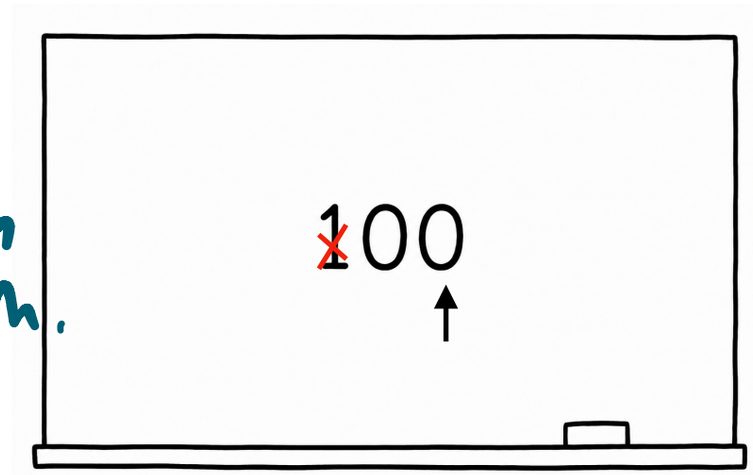
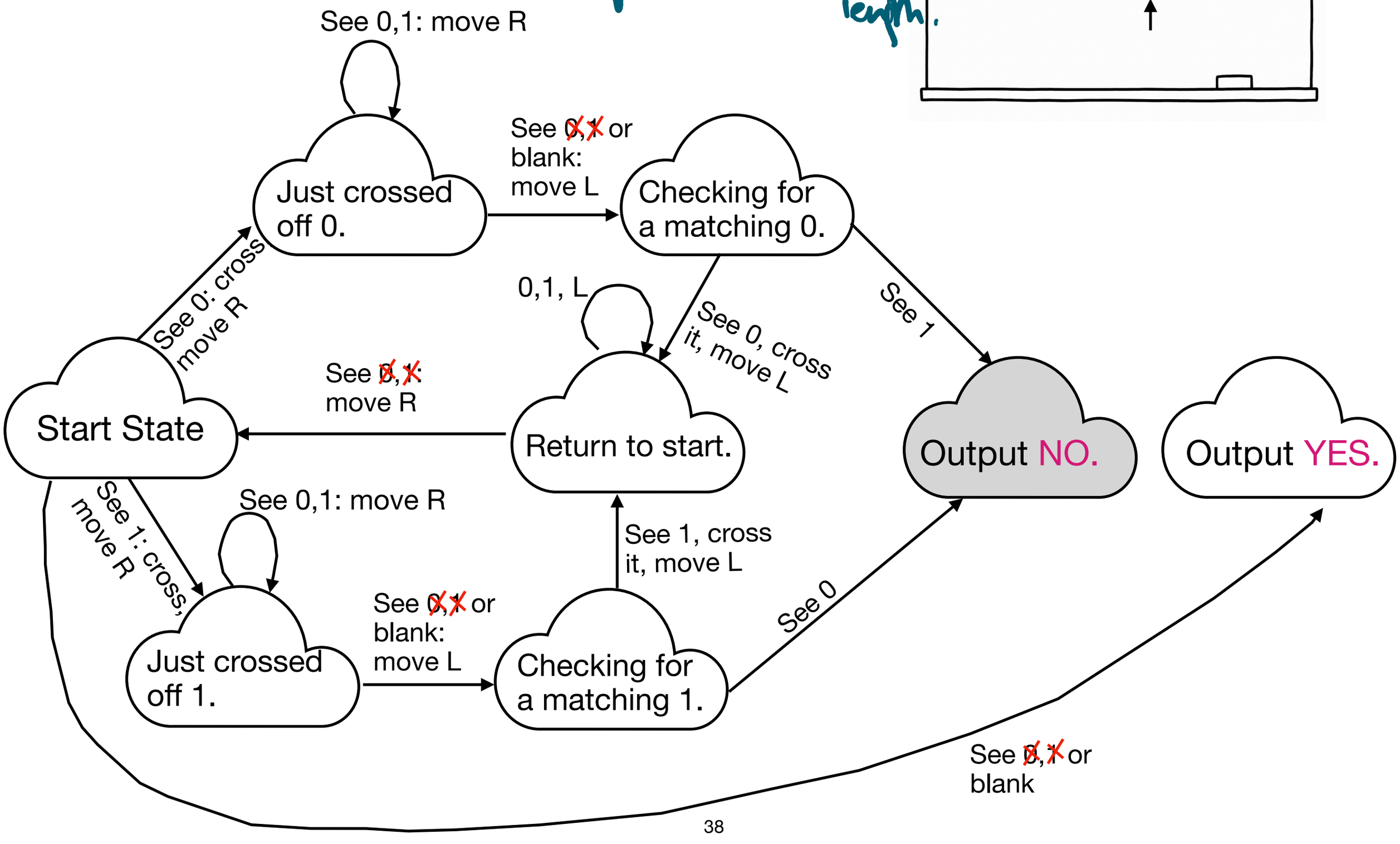

See 0,1: move R
Just crossed off 0.
See 0, 1 or blank: move L
Checking for a matching 0.
0,1, L
See 0, cross it, move L
See 0: cross, move R
See 0, 1: move R
Start State
Return to start.
Output NO.
Output YES.
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
See 0, 1 or blank: move L
Just crossed off 1.
Checking for a matching 1.
See 0
See 0, 1 or blank

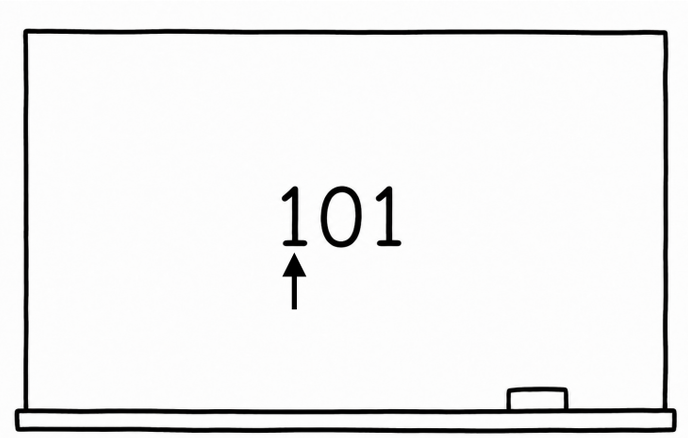
Examples

1) ε
2) non-palindromes
3) palindromes of even length.





Examples



See 0,1: move R

Just crossed off 0.

See ~~0~~,~~1~~ or blank: move L

Checking for a matching 0.

0,1, L

See 0, cross it, move L

See 1

See 0: cross, move R

See ~~0~~,~~1~~: move R

Start State

Return to start.

Output NO.

Output YES.

See 1: cross, move R

See 0,1: move R

See 1, cross it, move L

See ~~0~~,~~1~~ or blank: move L

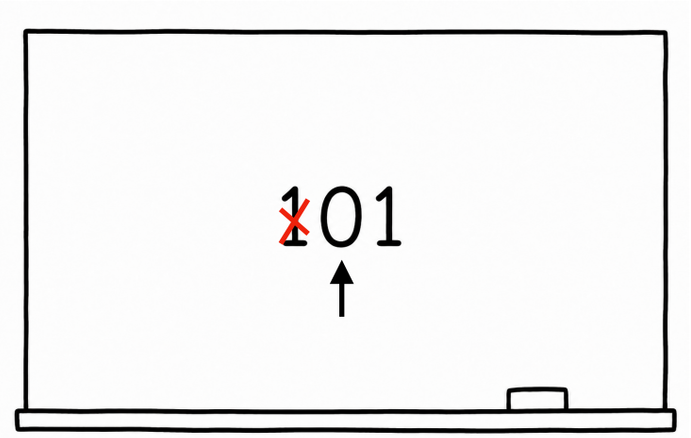
Just crossed off 1.

Checking for a matching 1.

See 0

See ~~0~~,~~1~~ or blank

Examples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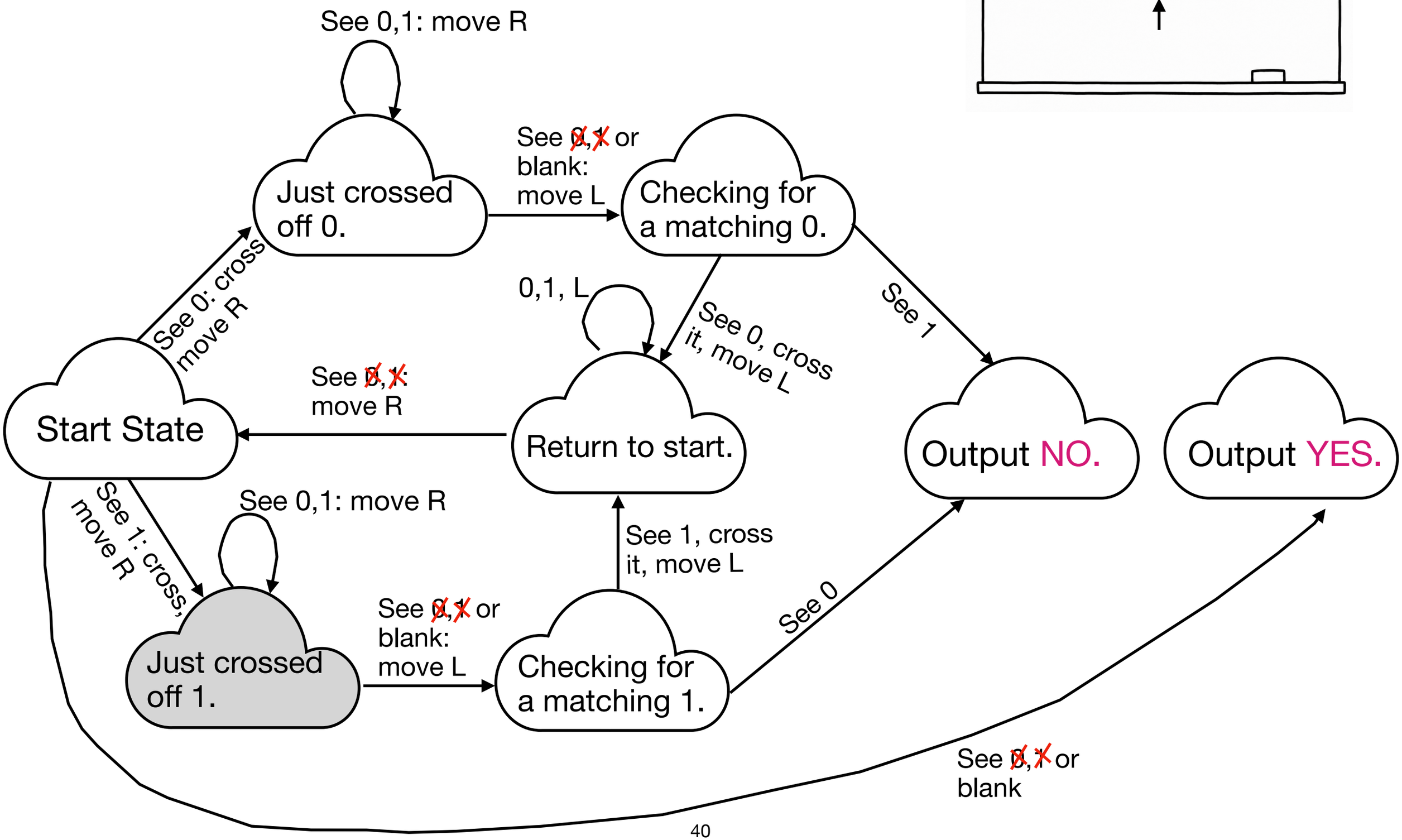
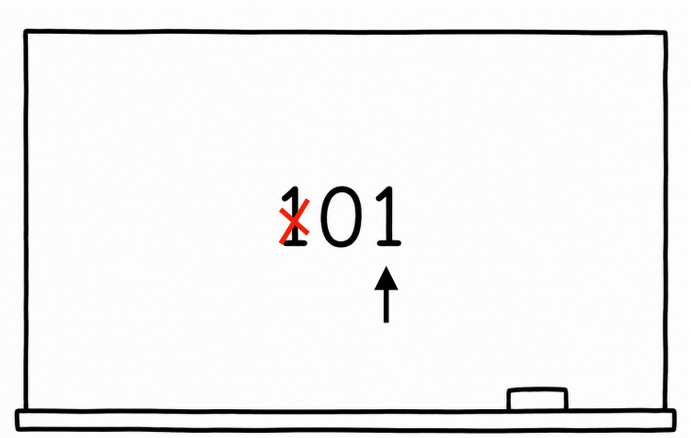

See 0,1: move R
Just crossed off 0.
See 0 or 1 or blank: move L
Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
See 1
See 0 or 1: move R
Start State
Return to start.
Output NO.
Output YES.
See 1: cross it, move R
See 0,1: move R
See 1, cross it, move L
See 0 or 1 or blank: move L
Just crossed off 1.
Checking for a matching 1.
See 0
See 0 or 1 or blank

Examples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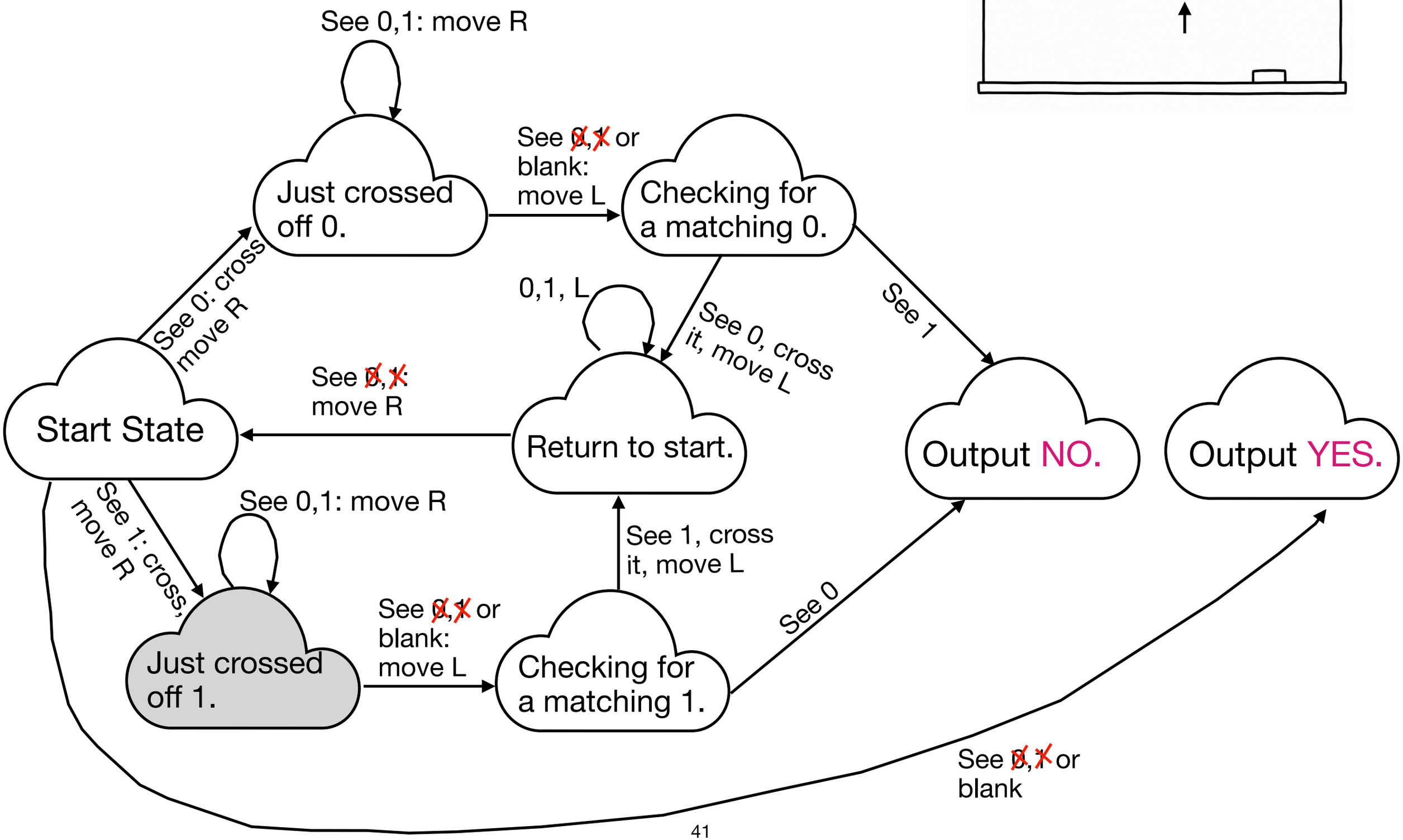
See 0,1: move R
Just crossed off 0.
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Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
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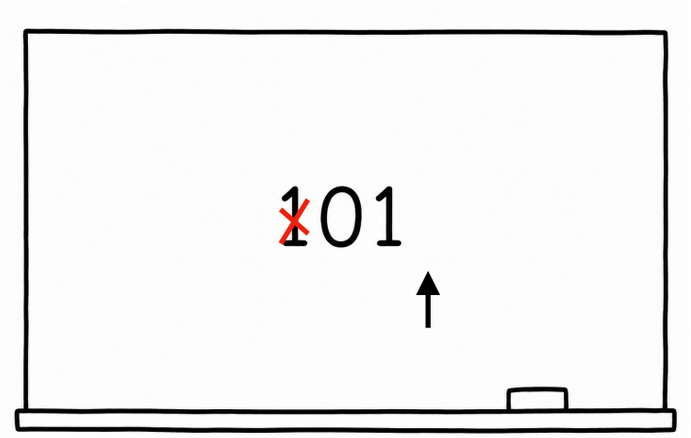
Return to start.
Output NO.
Output YES.
See 1: cross it, move R
See 0,1: move R
See 1, cross it, move L
See 0, 1 or blank: move L
Just crossed off 1.
Checking for a matching 1.
See 0
See 0, 1 or blank

Examples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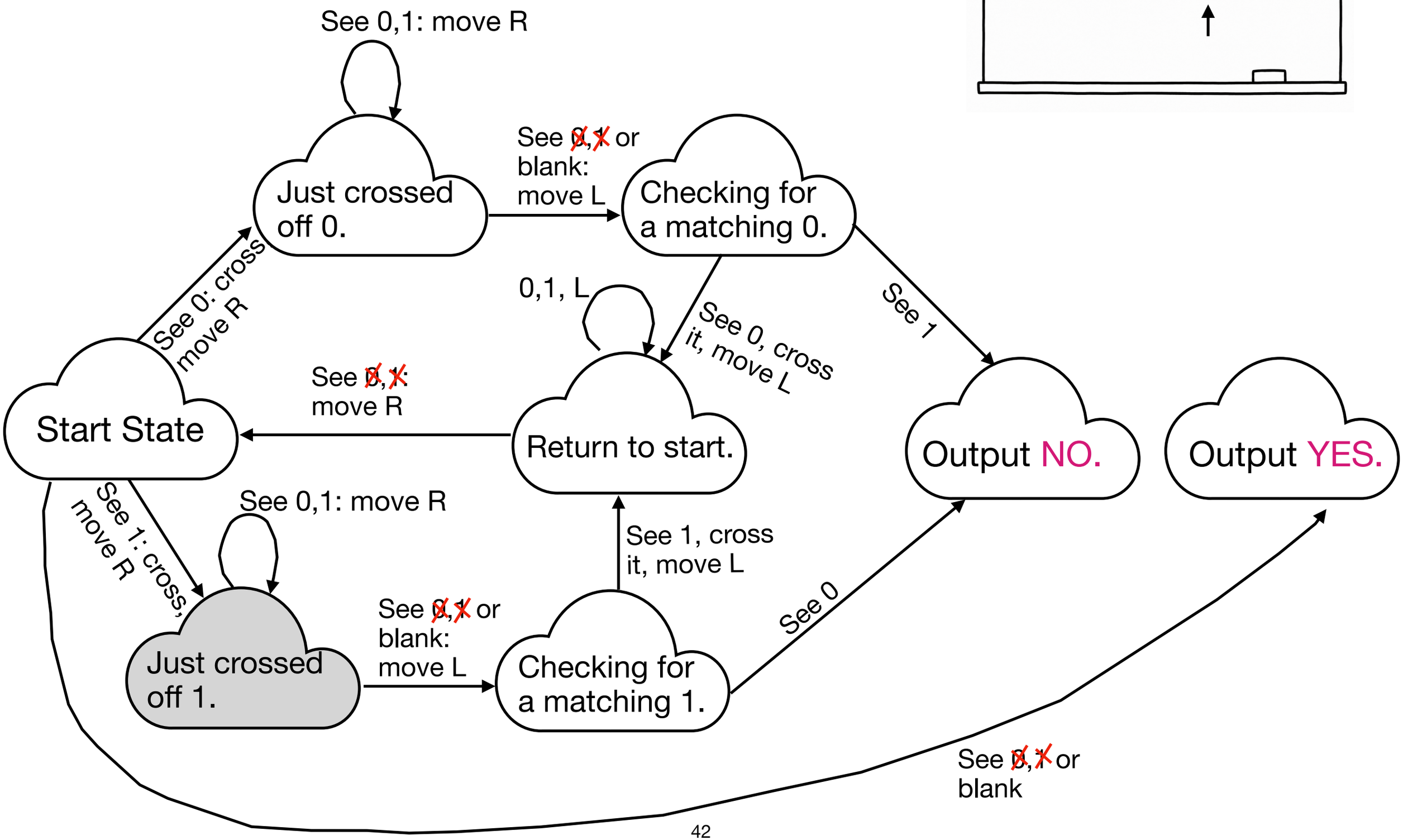
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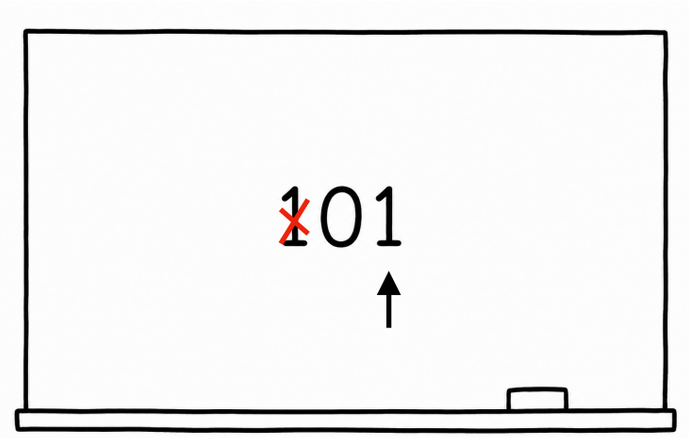
Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
See 1
See 0, 1: move R
Start State
Return to start.
Output NO.
Output YES.
See 1: cross it, move R
See 0,1: move R
See 1, cross it, move L
See 0, 1 or blank: move L
Just crossed off 1.
Checking for a matching 1.
See 0
See 0, 1 or blank

Examples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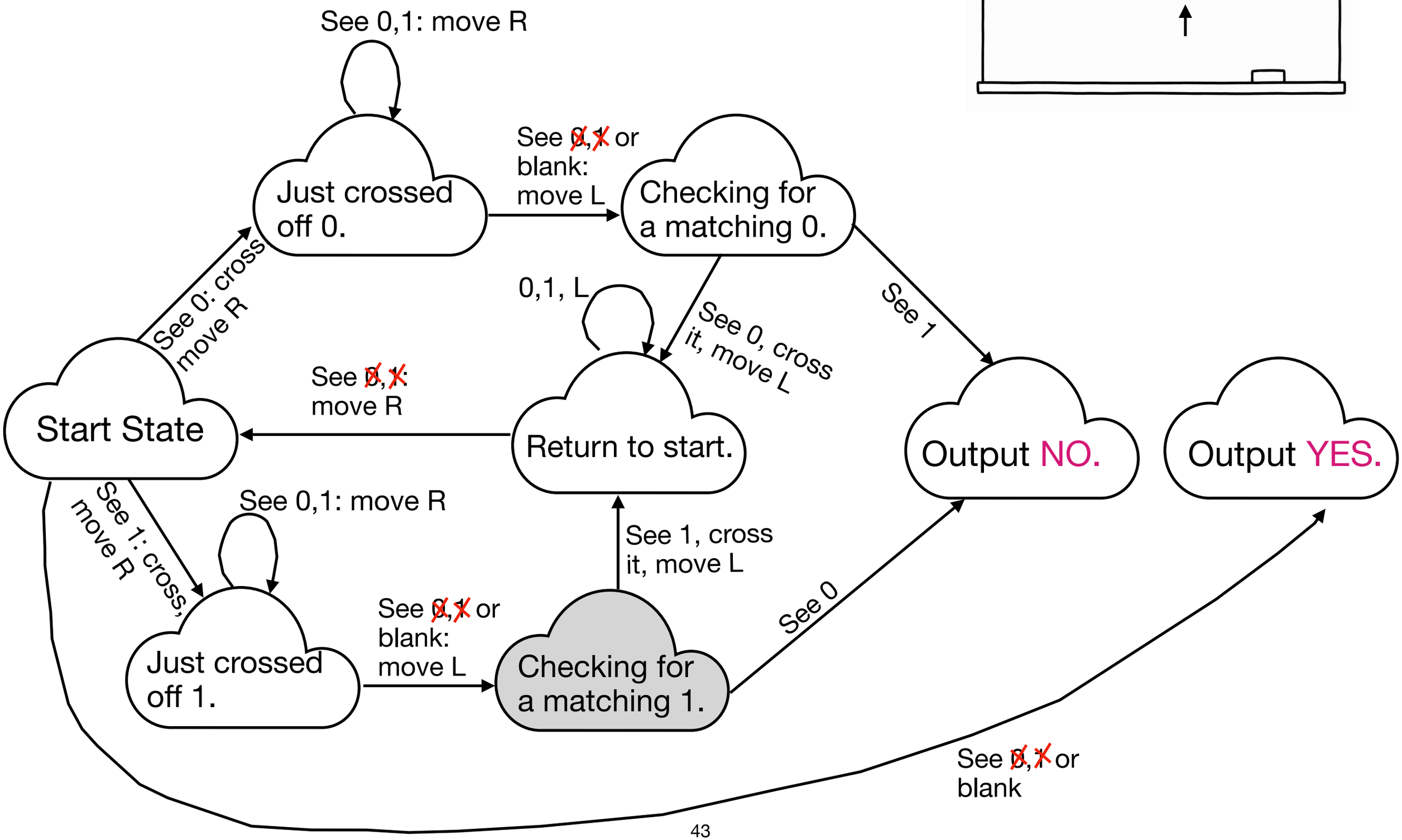
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Checking for a matching 0.
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0,1, L
See 0, cross it, move L
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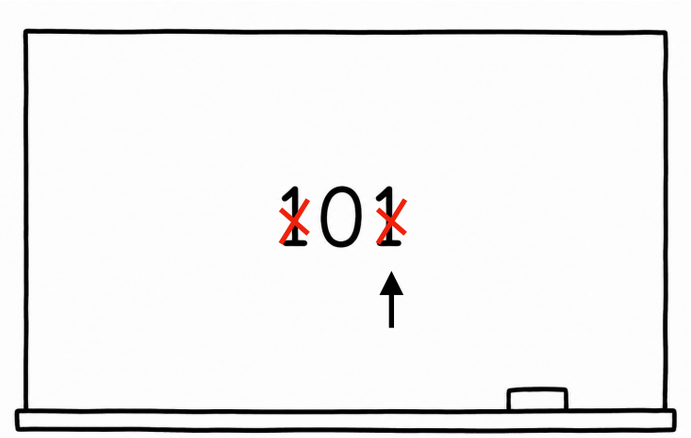
Just crossed off 1.
Checking for a matching 1.
See 0
See 0, 1 or blank

Examples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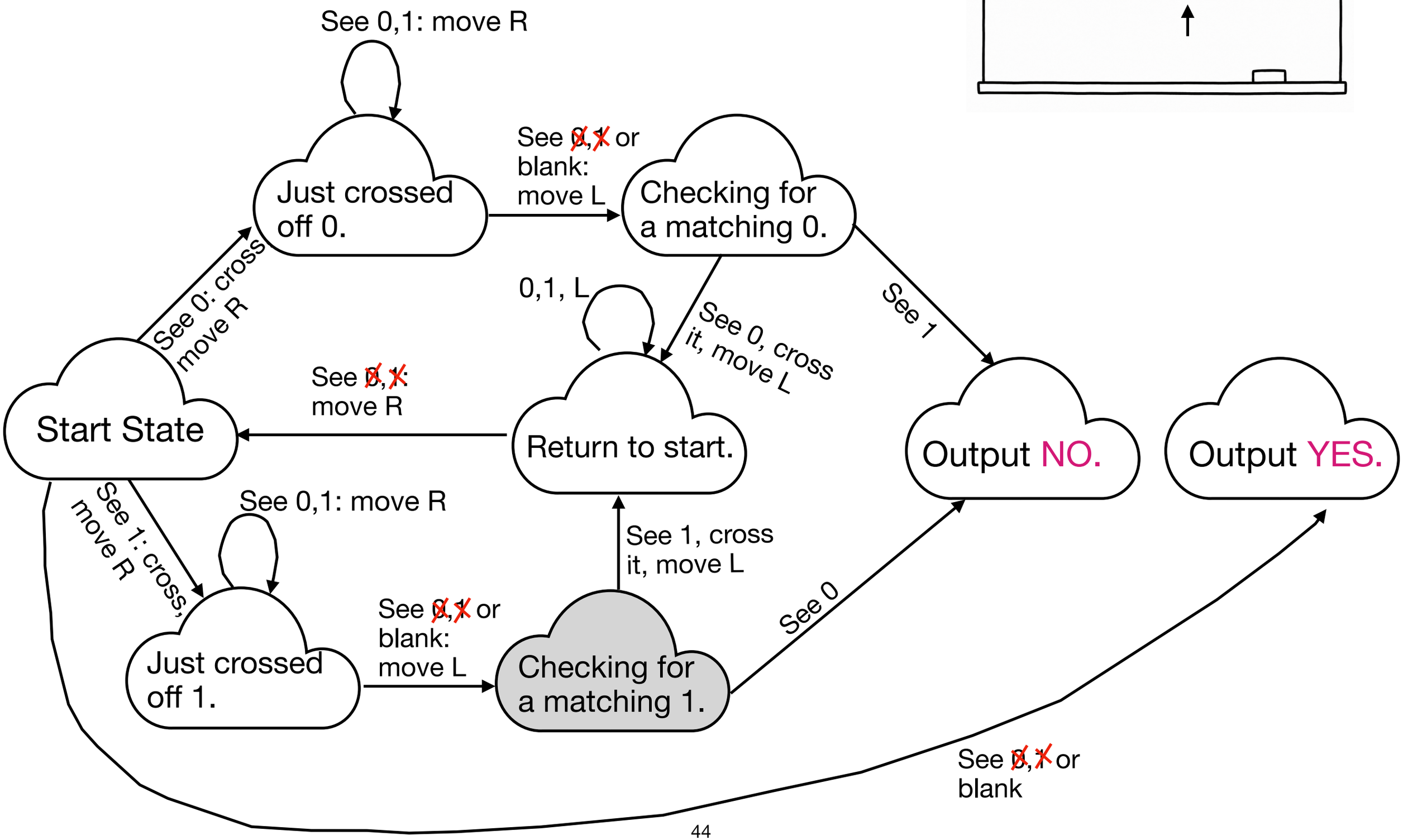
See 0,1: move R
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Checking for a matching 0.
See 0: cross, move R
0,1, L
See 0, cross it, move L
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See 0, 1: move R
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Return to start.
Output NO.
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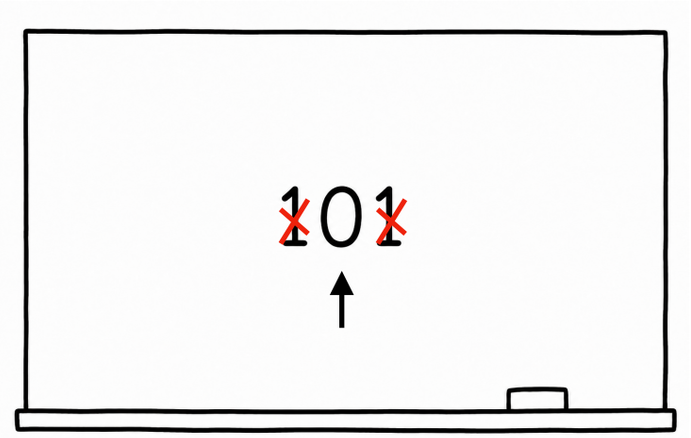
See 1: cross, move R
See 0,1: move R
See 1, cross it, move L
See 0, 1 or blank: move L
Just crossed off 1.
Checking for a matching 1.
See 0
See 0, 1 or blank

Examples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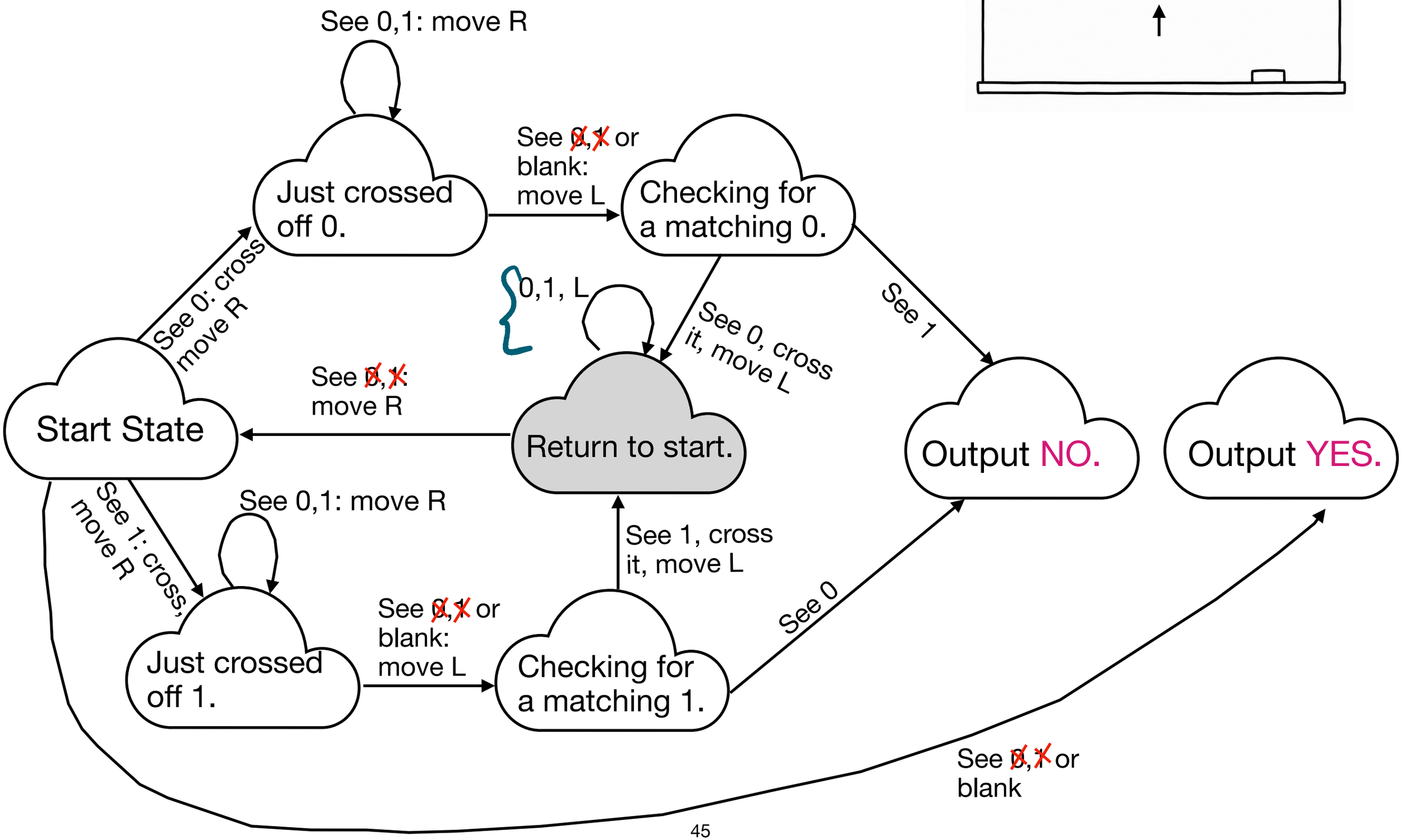
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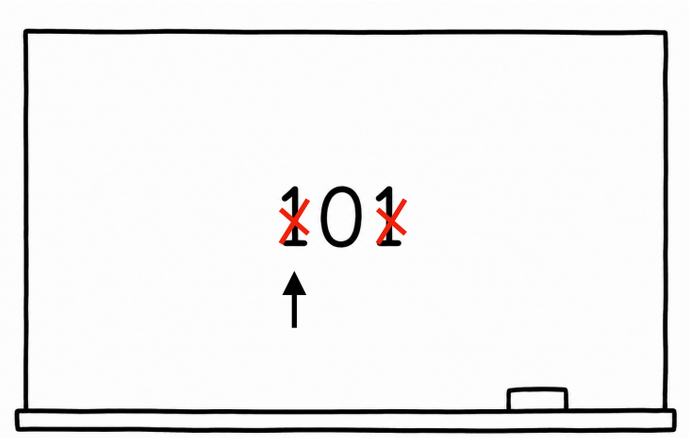
Examples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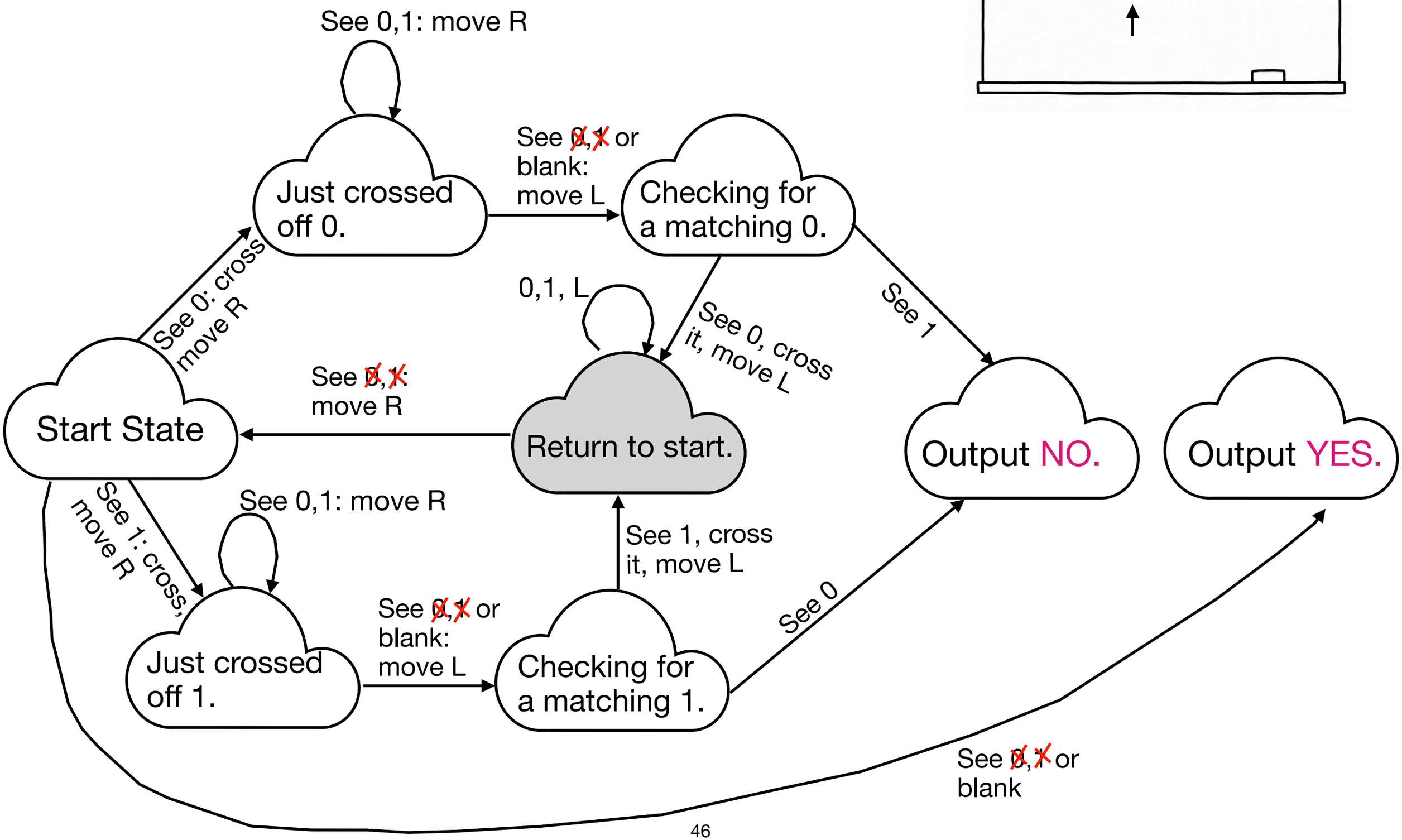
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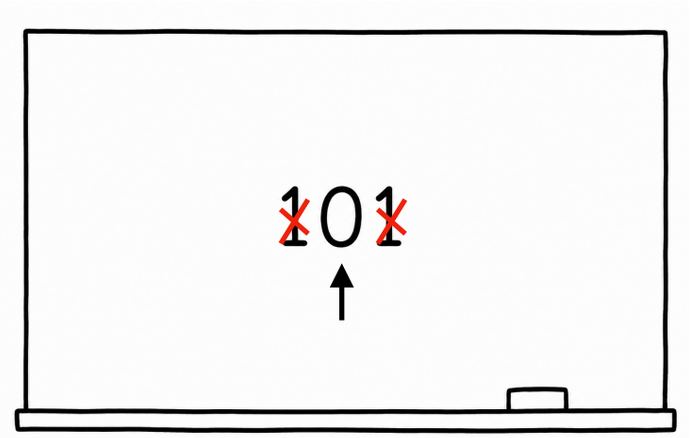
Examples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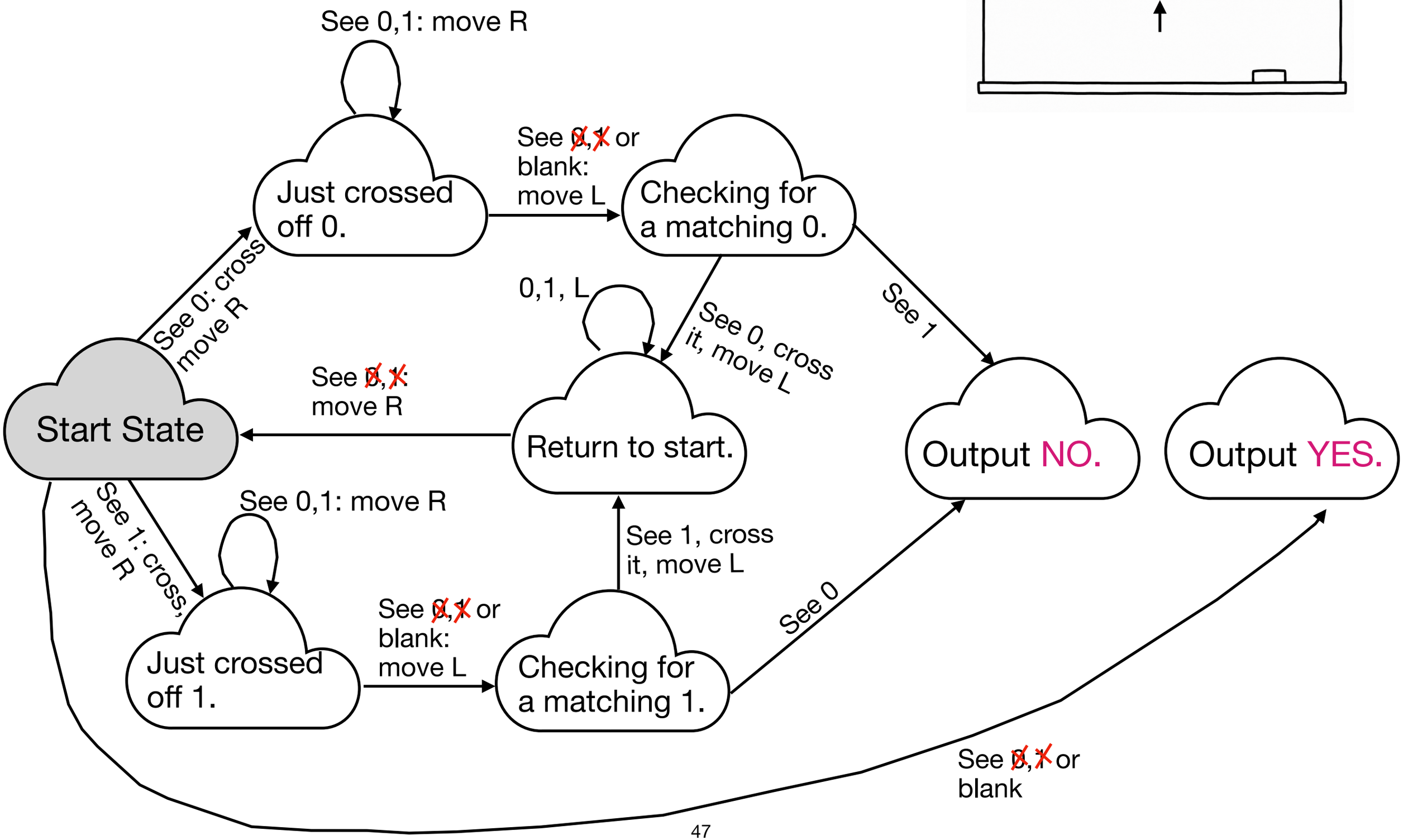
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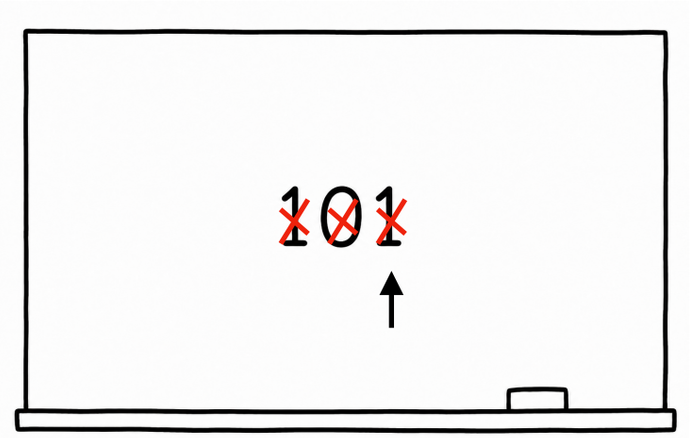
Examples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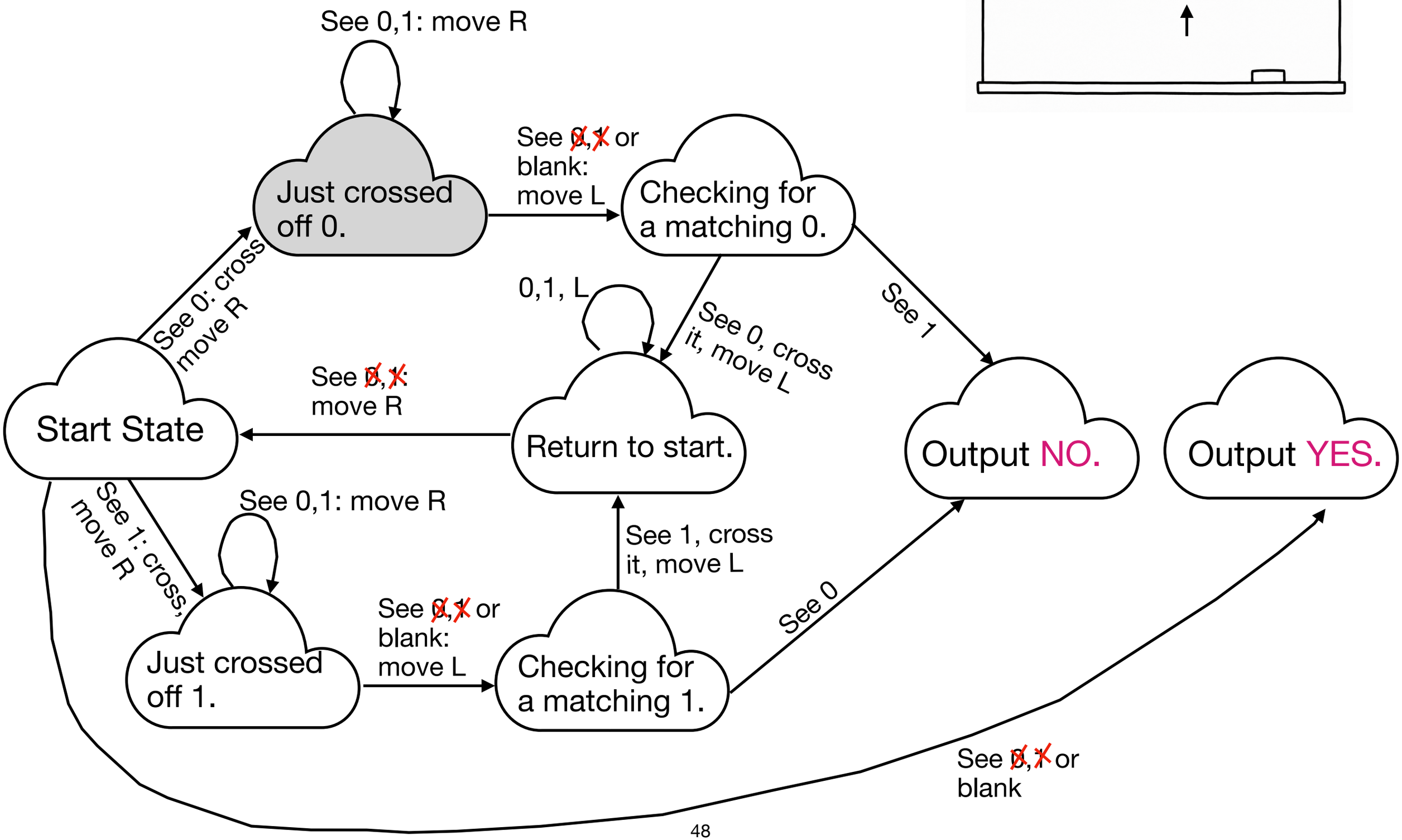
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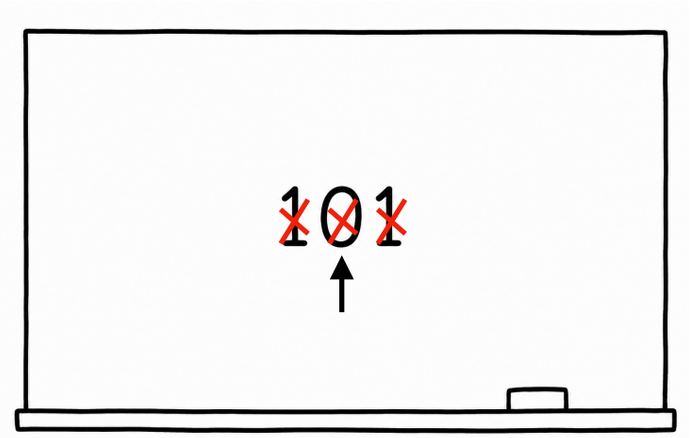
See 0, 1 or blank

Examples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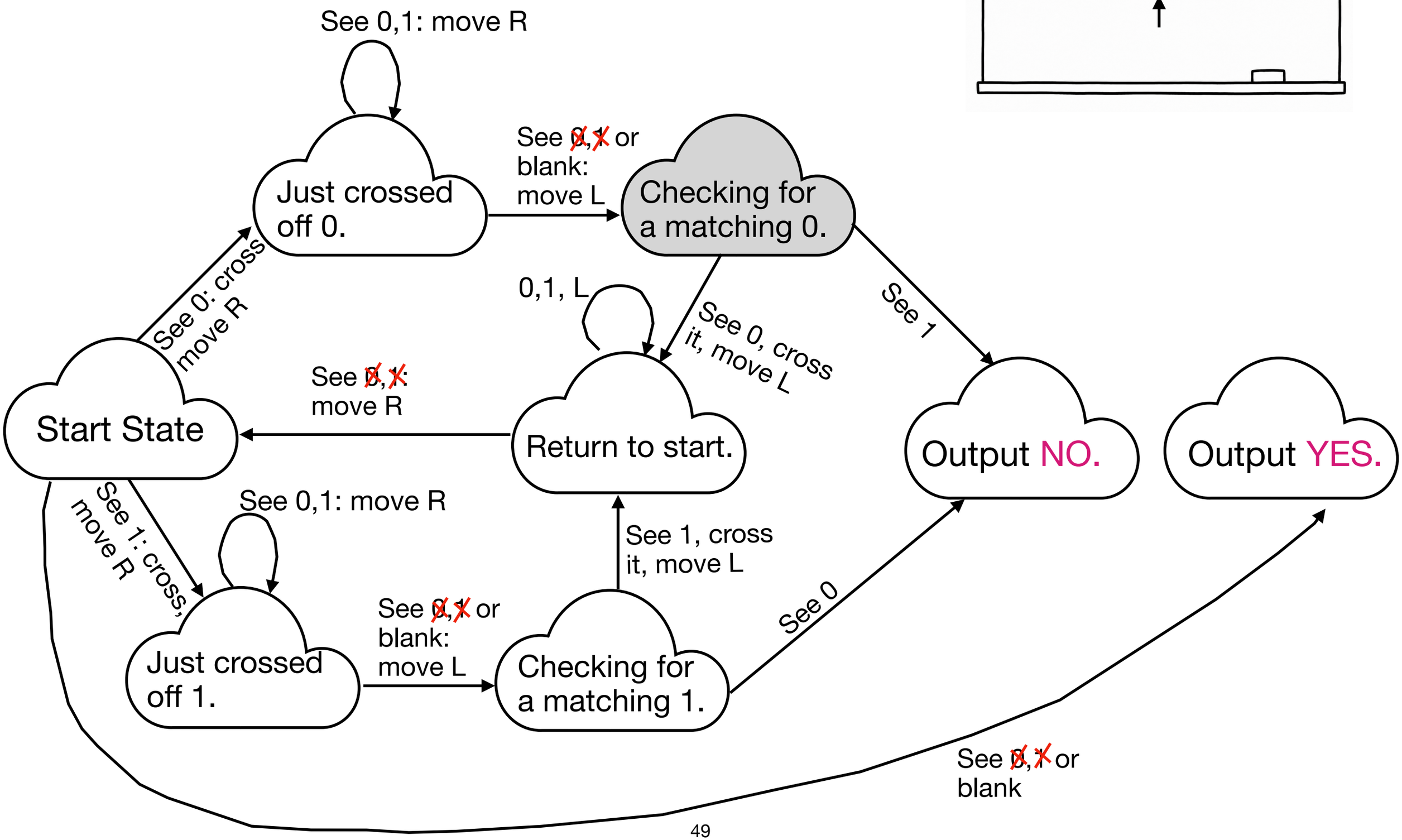
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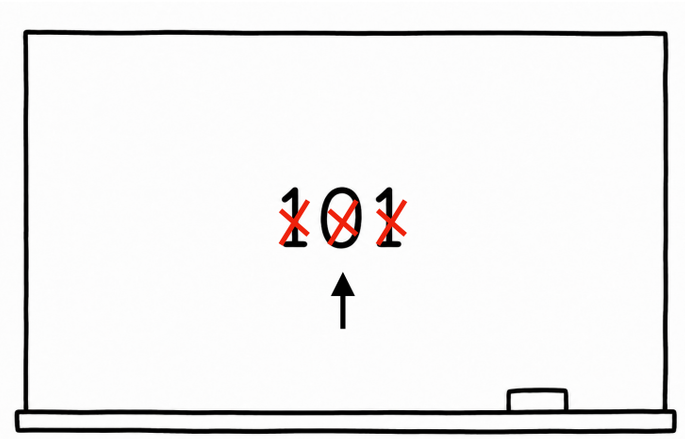
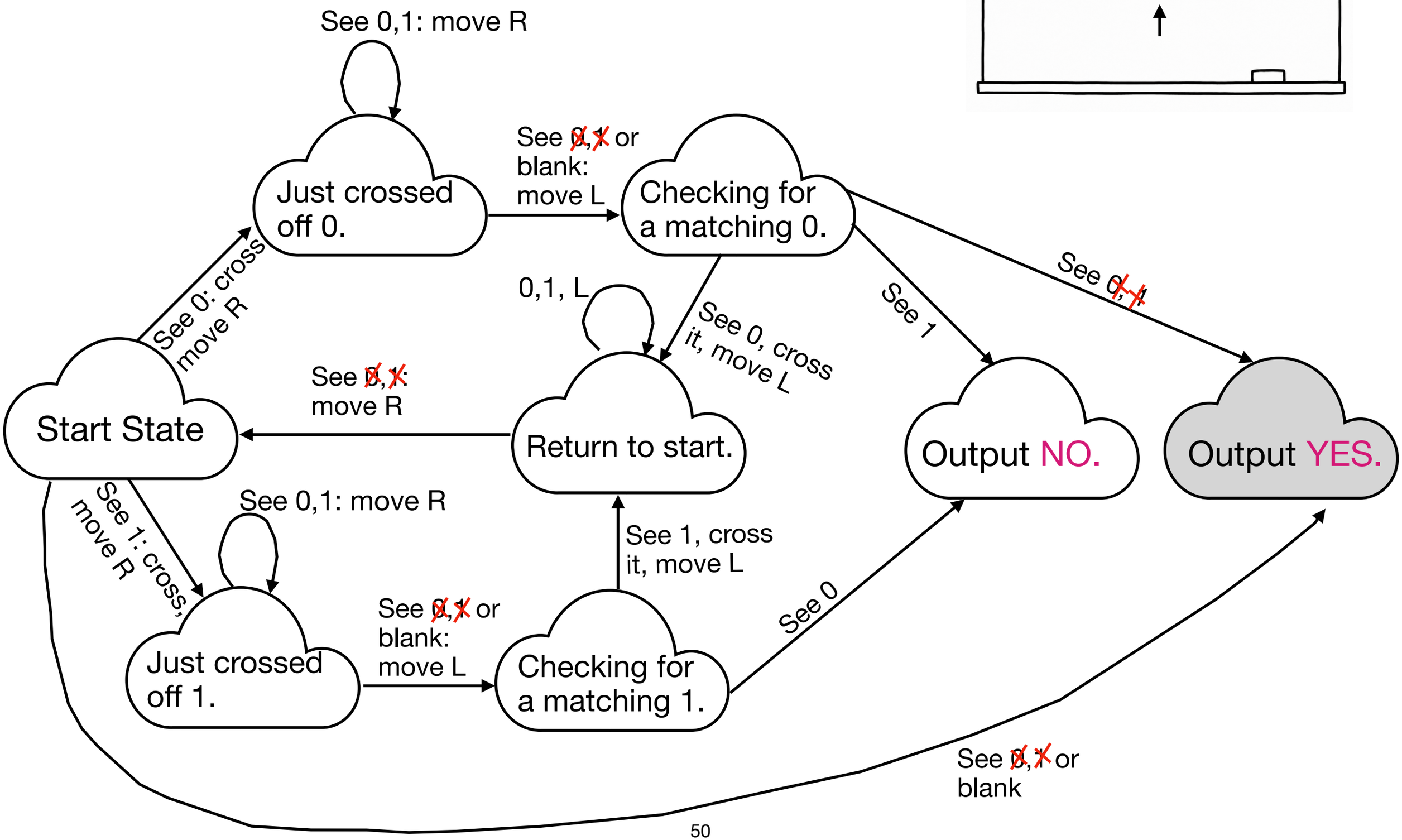
Checking for a matching 1.
See 0
See 0, 1 or blank

Examples


101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Checking for a matching 0.
See 0: cross it, move R
0,1, L
See 0, cross it, move L
See 1
See 0, 1
See 0, 1: move R
Start State
Return to start.
Output NO.
Output YES.
See 1: cross, move R
See 1, cross it, move L
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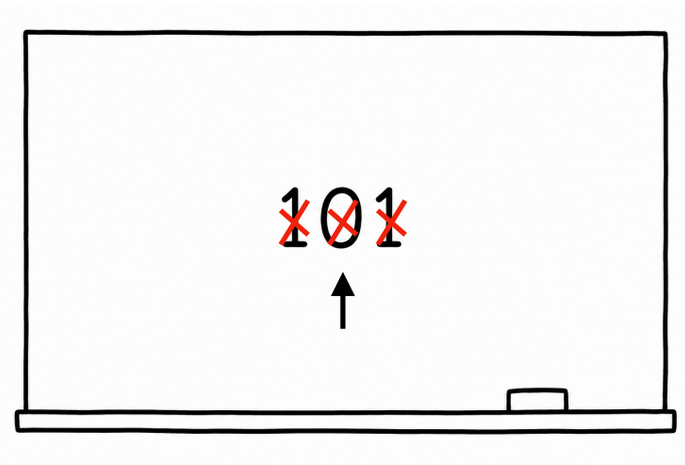
Examples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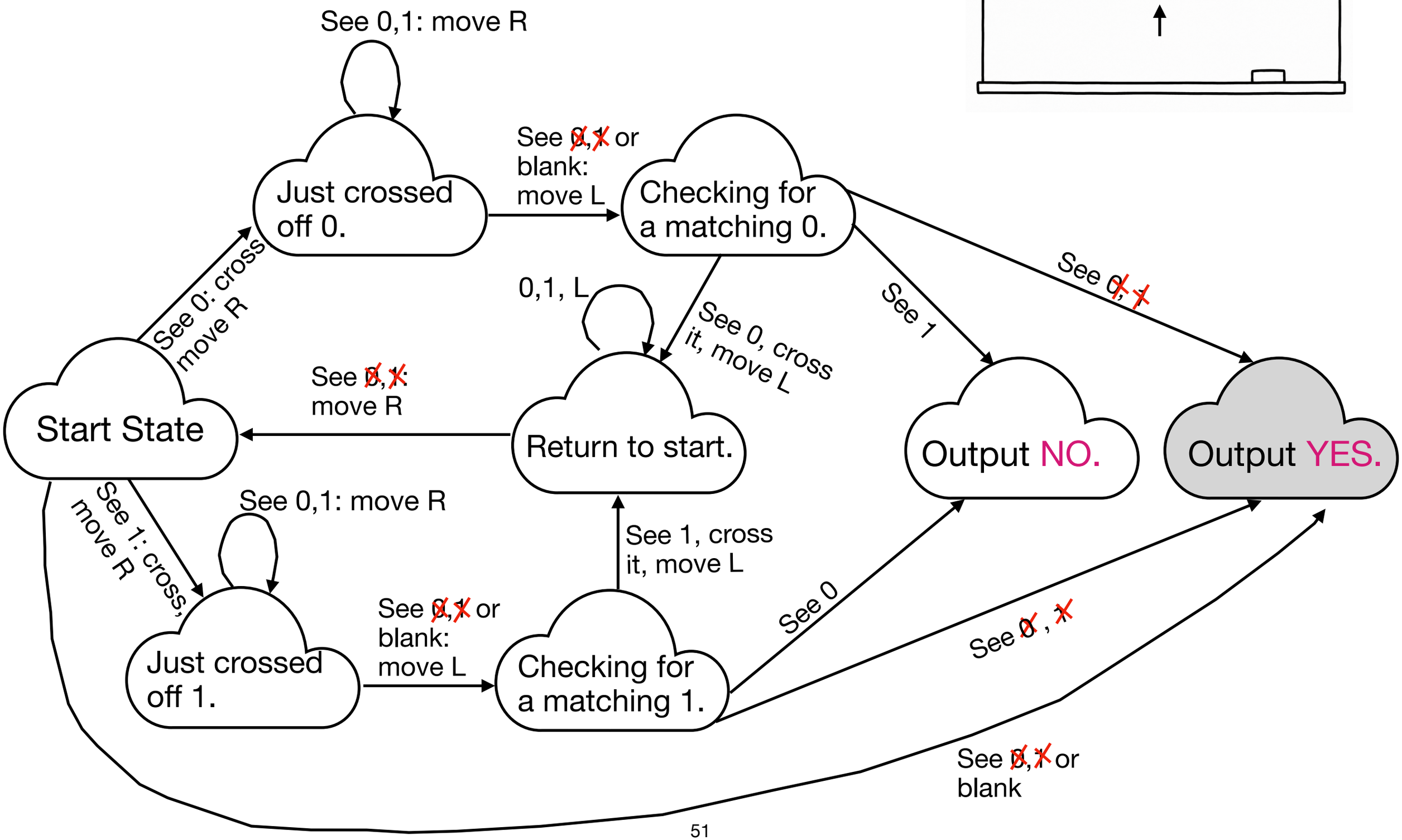
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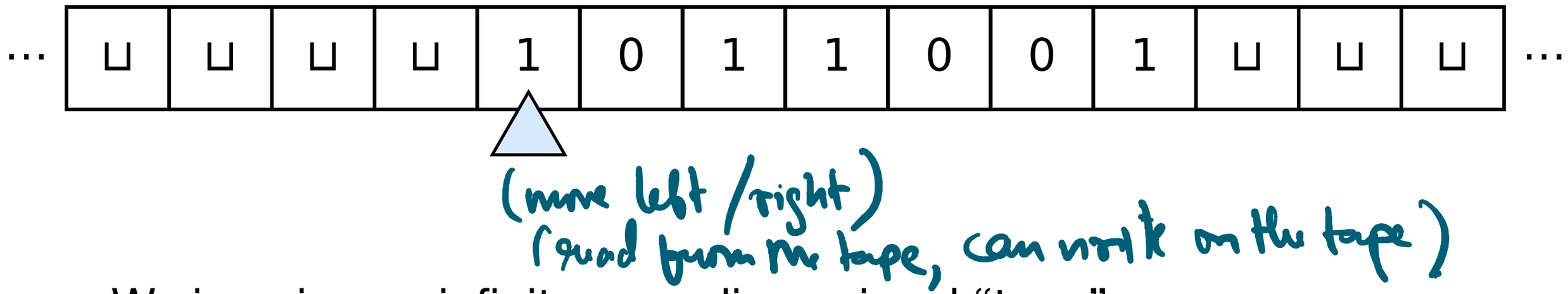
The Turing Machine Model

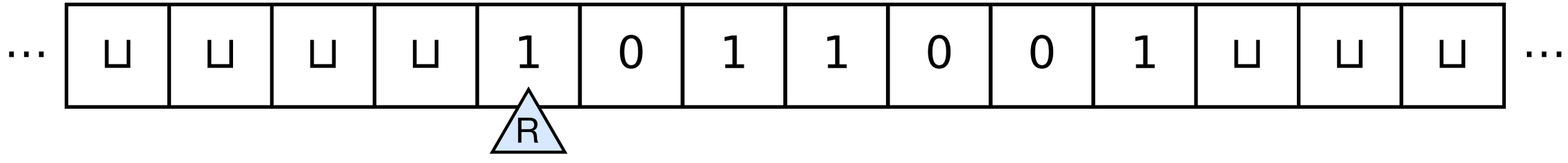
- Turing Machines: A mathematical model of human computation.
- The state diagram illustrates the local, step-by-step behavior that we will formalize as a Turing Machine.

The Turing Machine Model



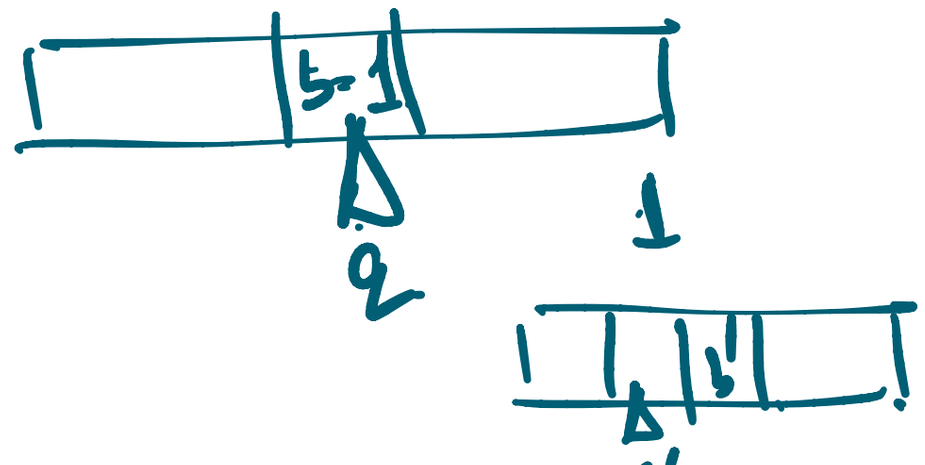
- We imagine an infinite, one-dimensional “tape”.
- The tape divided into “cells”.
- Each cell contains a single symbol.
- There is a “head” pointing at one cell of the tape.
- The machine can be in one of finitely many “internal states”

The Turing Machine Model



- In each step, the machine decides:
  - What to write in current cell,
  - Which cell to move in (left/right),
  - And, the next state.
- The decision is based only on the symbol being read and the current state the machine is in.

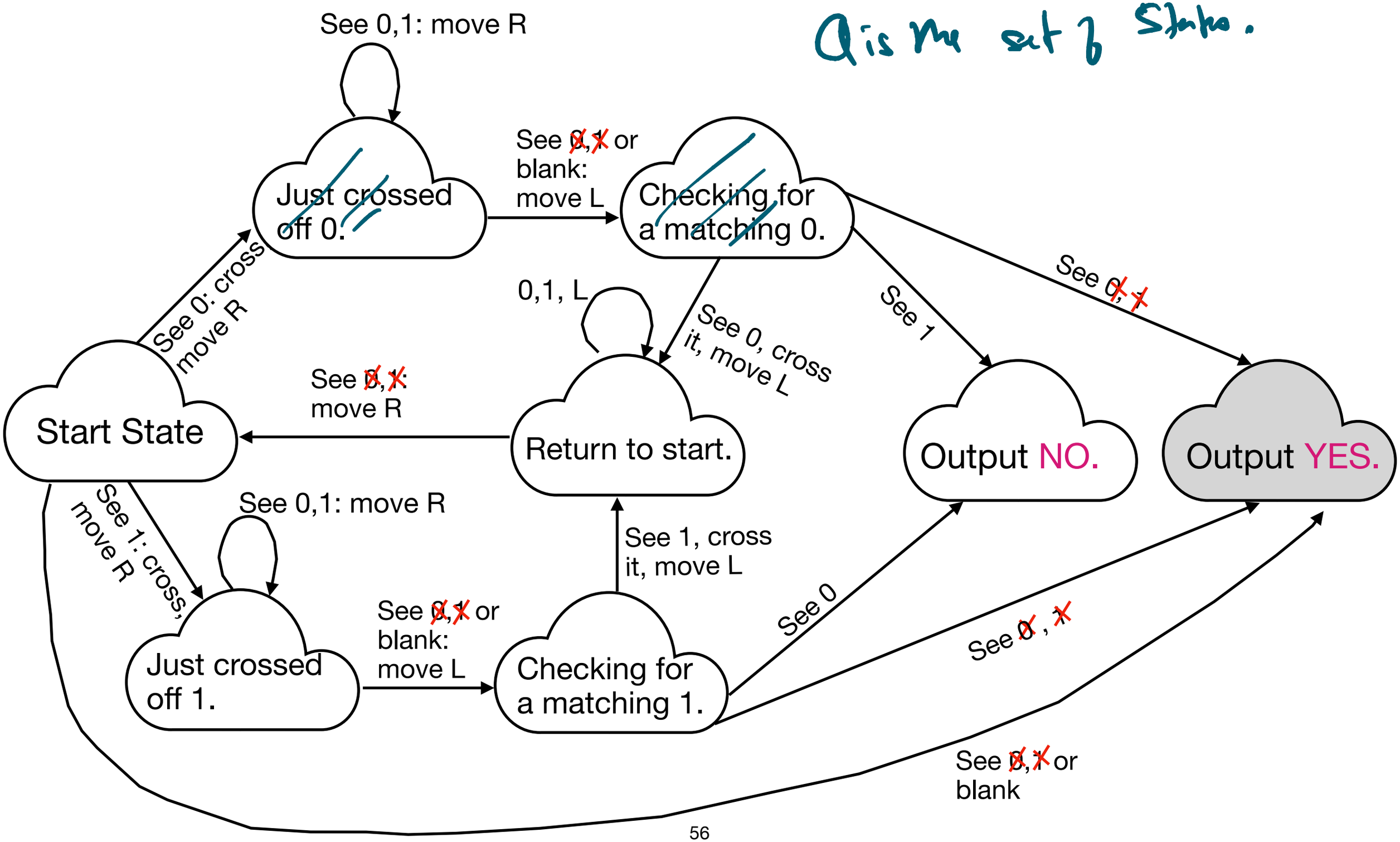
Transition Function



- Mathematically, a Turing machine is described by a transition function: $\delta : Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$.
- Here Q is the set of states and Σ is the set of symbols.
- $\delta(q, b) = (q', b', D)$ means:
  - If we are in state q and read the symbol b.
  - Write b' (replacing b), move the head in direction D (L = left and R = right), and go to state q'.

Examples

Q is the set of States.



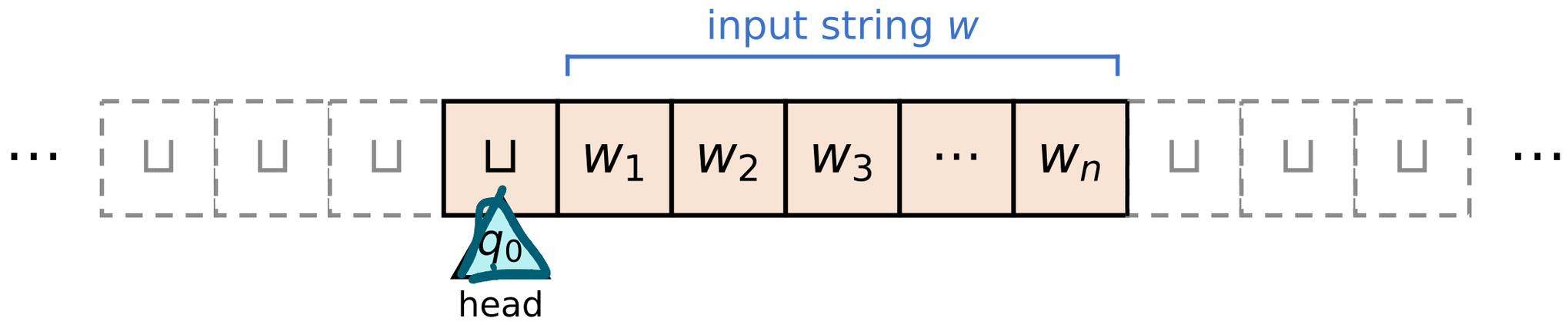
Input

- One Turing Machine represents one algorithm.
- For us, the input to a Turing machine will always be a string over the binary alphabet.

$\{0,1,\sqcup\} \subseteq \Sigma$ = tape alphabet

$\not{\sqcup}, x \in \Sigma$

Turing Machine Initialization



- The tape initially contains an input string $w \in \{0,1\}^*$.
- The head is initially in a cell to the left of the input.
  - The cell contains a special "blank symbol" $\sqcup$.
- The machine is initially in a special "start state" q_0.

Halting

$\delta(q_0, \sqcup) \to (q_1, \sqcup, R)$

- After initialization, we repeatedly apply the transition function.
- Eventually, the computation may reach a "halting state":
  - There are two halting states: q_{accept} and q_{reject}.
- In this case, the computation halts.

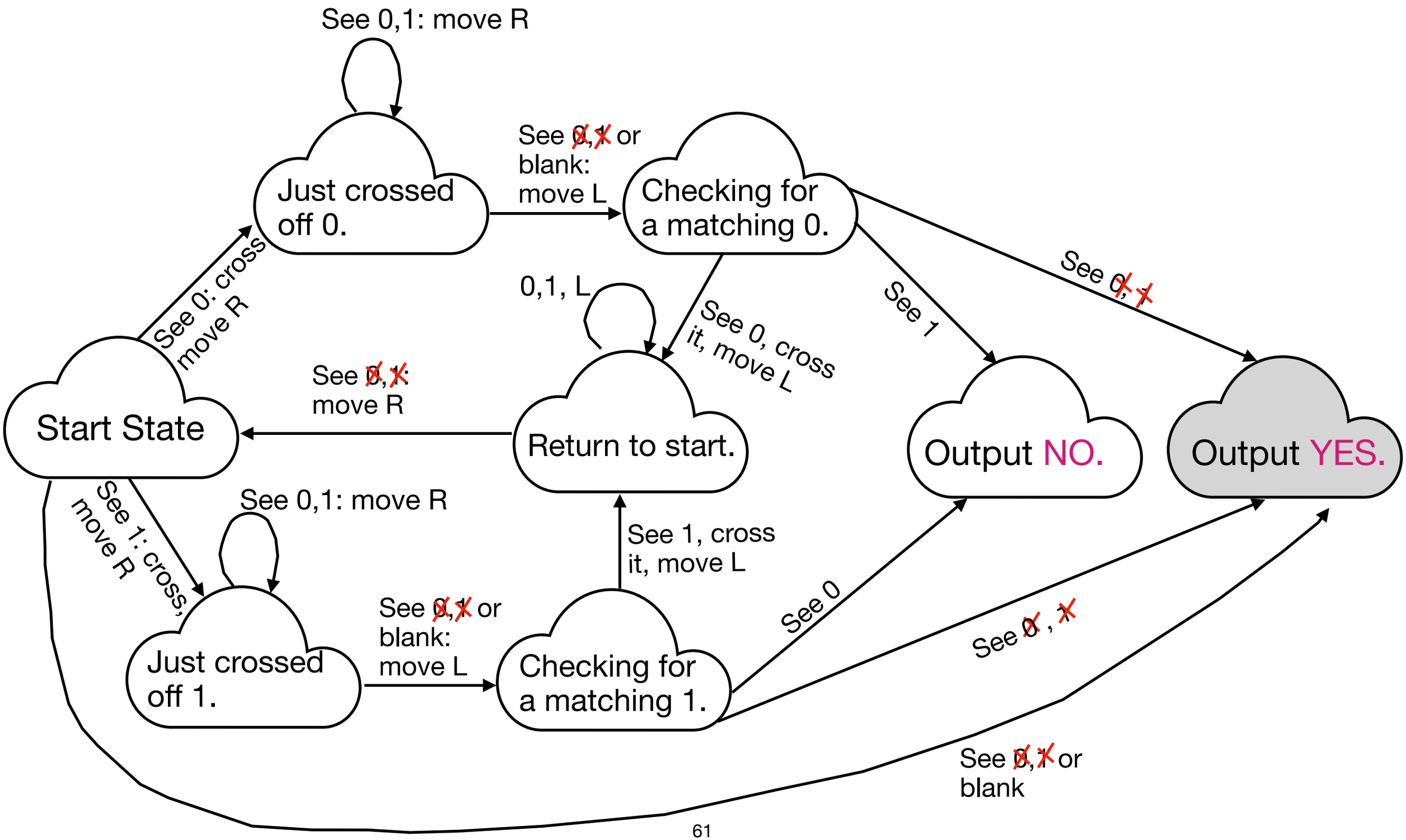
$\Downarrow$ output YES

$\Downarrow$ output NO.

Accepting and Rejecting

- If the machine enters q_{accept}, we say it accepts the input.
  - Analogy: "`return True`".
- If the machine enters q_{reject}, we say it rejects the input.
  - Analogy: "`return False`".

Examples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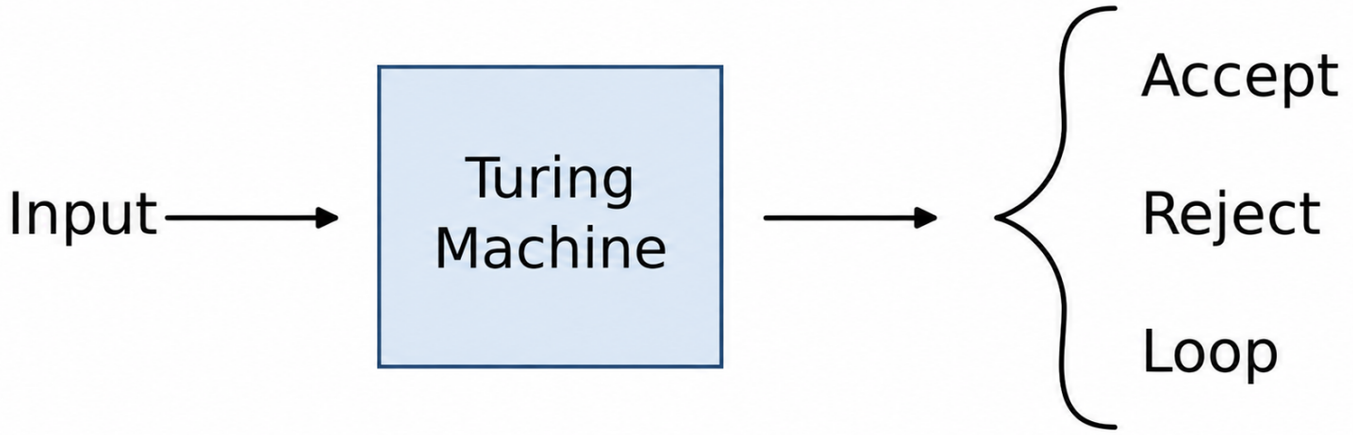
Looping

- It is also possible the machine runs forever without halting.
- In this case, we say the machine "loops".



Which problems can be solved through computation?

Deciding a Language

- Recall: In Lecture 1, decision problems = languages.
- Let M be a Turing Machine, and L be a language.
- We say M decides L if (M halts on all inputs)
  - M accepts every $x \in L$, and
  - M rejects every $x \notin L$ $\Rightarrow x \in \{0,1\}^* \setminus L$.
- This is a mathematical model of what it means to "solve a decision problem".

Consider a decision problem D, is it solvable?

(1) Language L_D corresponding to D.

(2) Is there a TM deciding L_D?

Yes then, D is solvable else it is not solvable.

Example: Parity

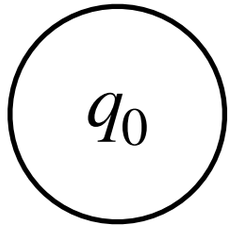
- **Informal Problem Statement:** "Given $w \in \{0,1\}^*$, determine whether the number of ones in w is even or odd".
- **Can be formulated as:**

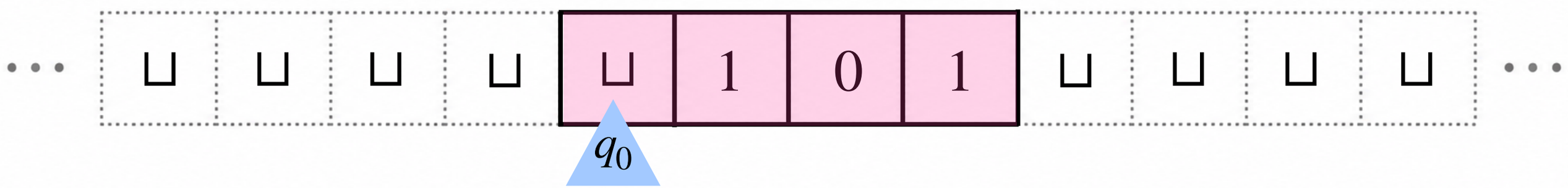
010 ∈ PARITY, 000 ∉ PARITY

PARITY $= \{w \in \{0,1\}^*$ such that w has an odd number of 1's$\}$

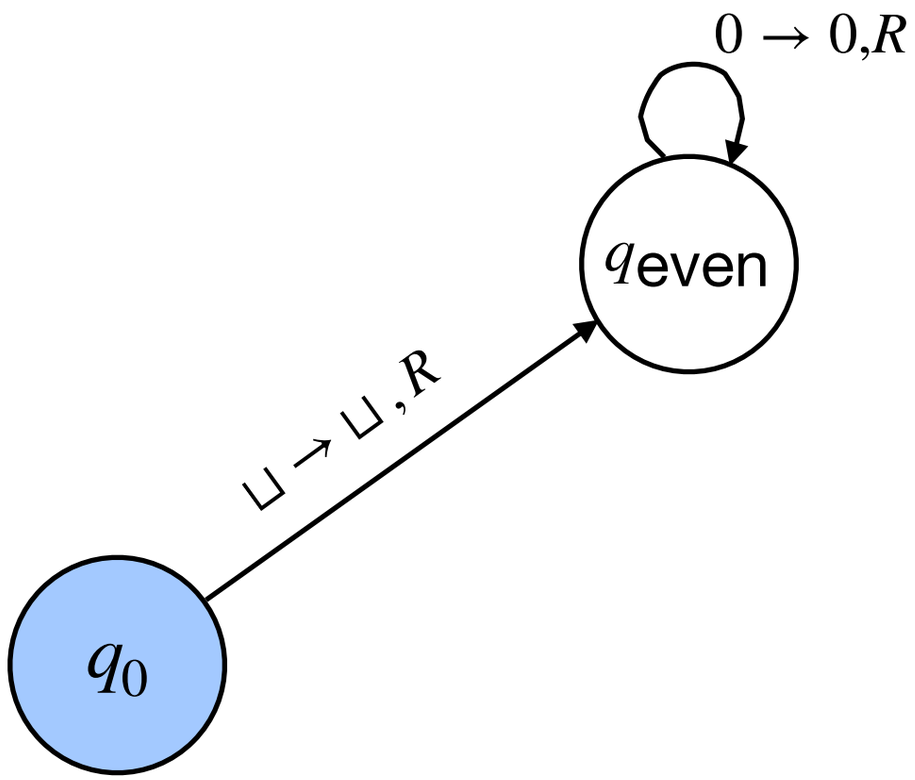
- Design a Turing Machine for PARITY.

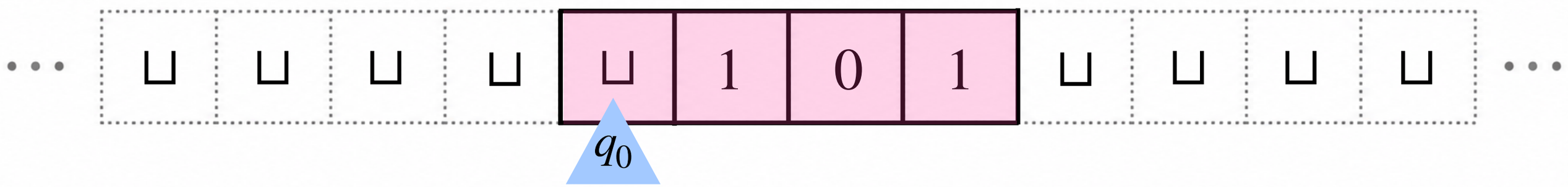
Example: Parity



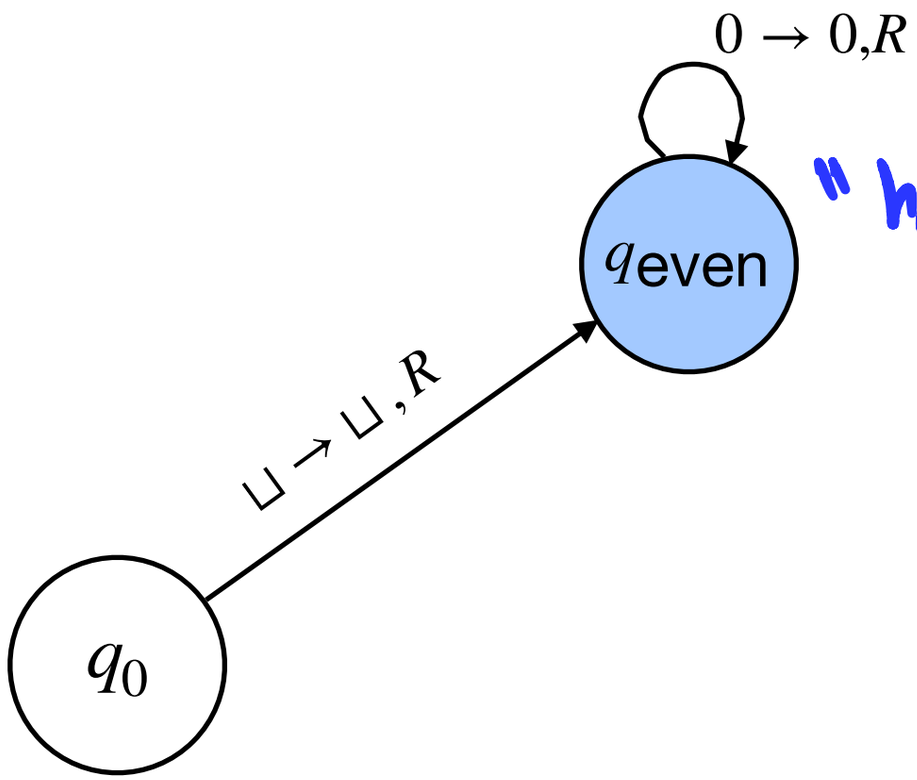


Example: Parity

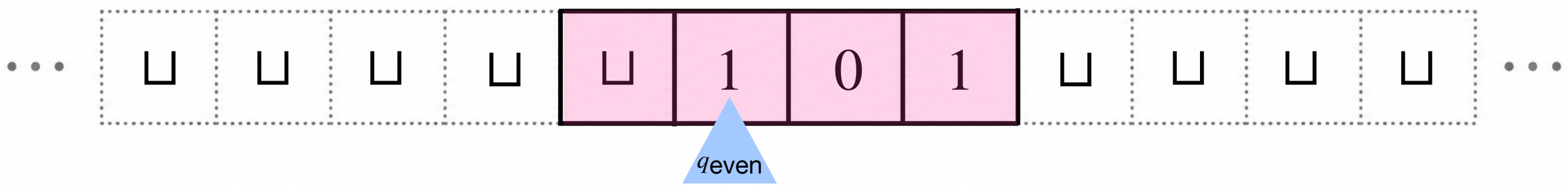




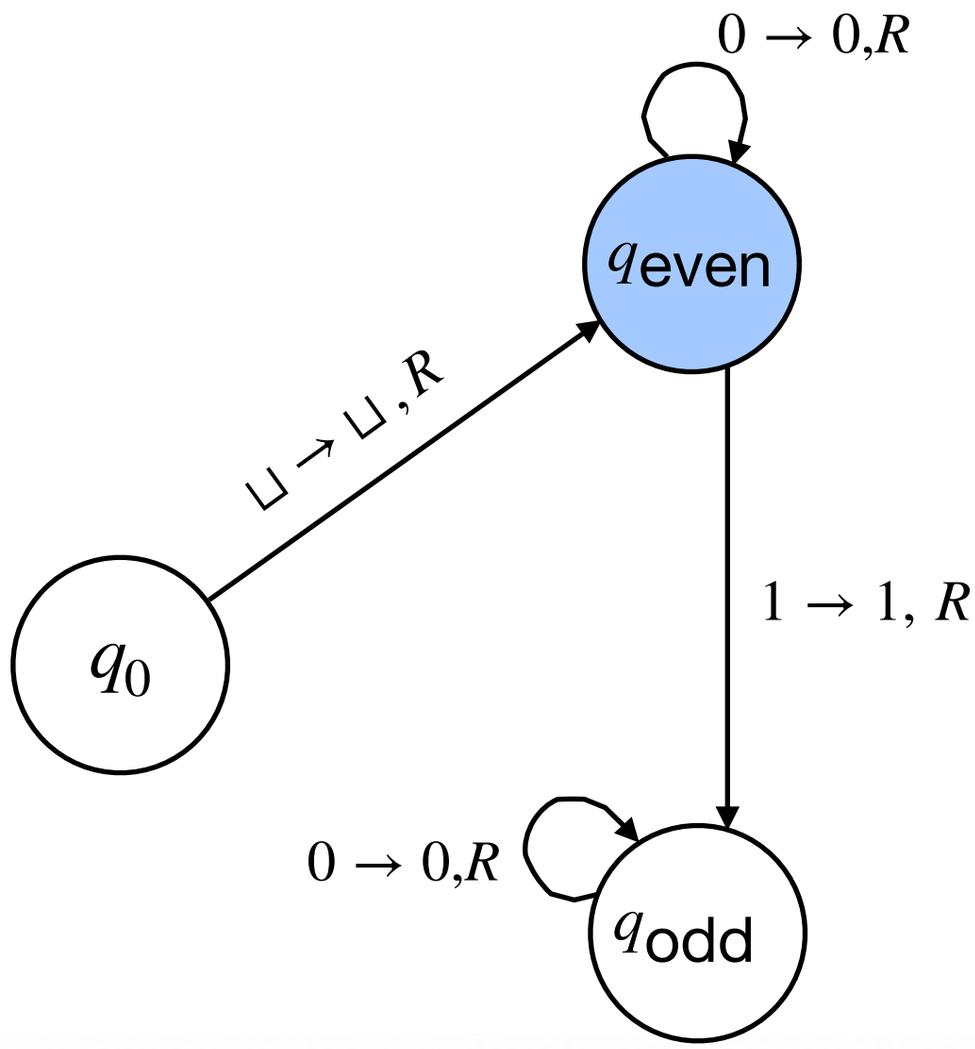
Example: Parity

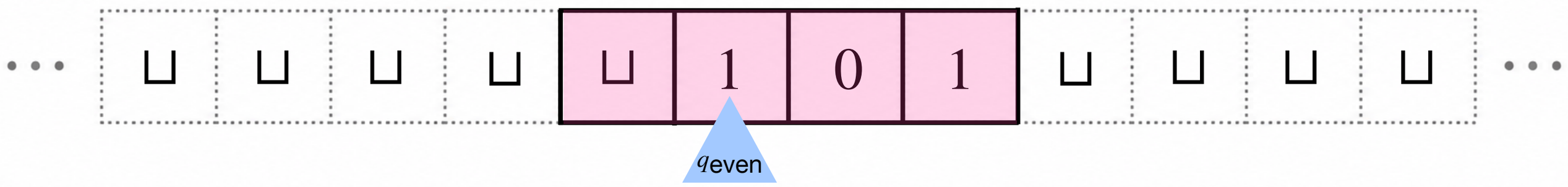


"have seen an even # of 1's",

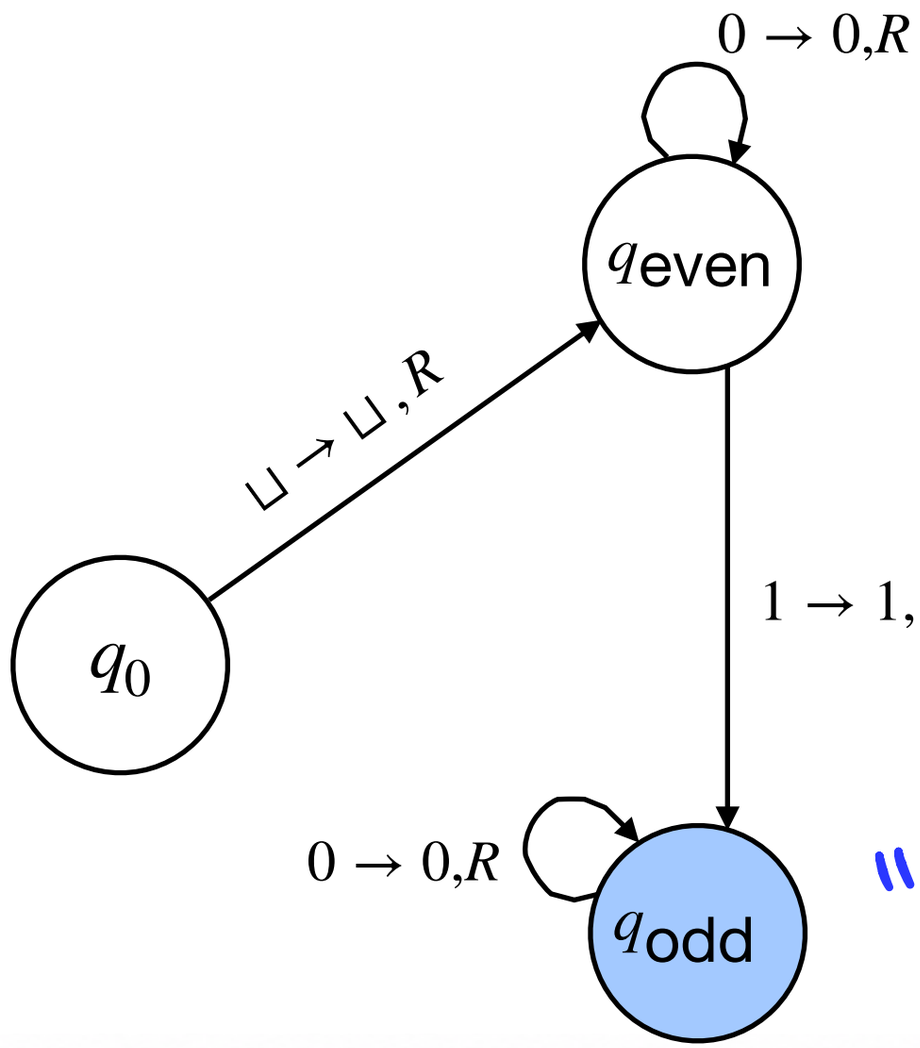


Example: Parity

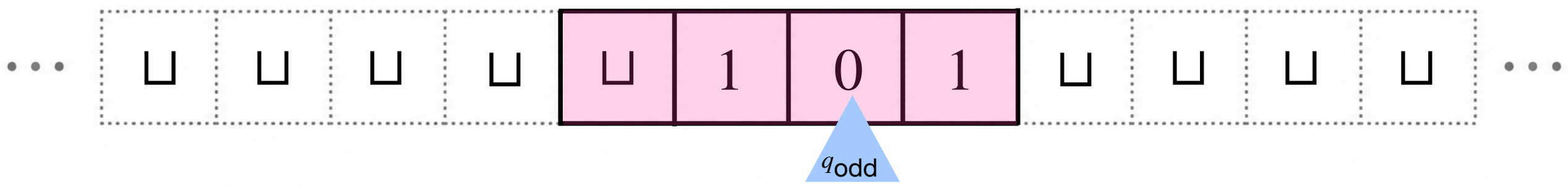




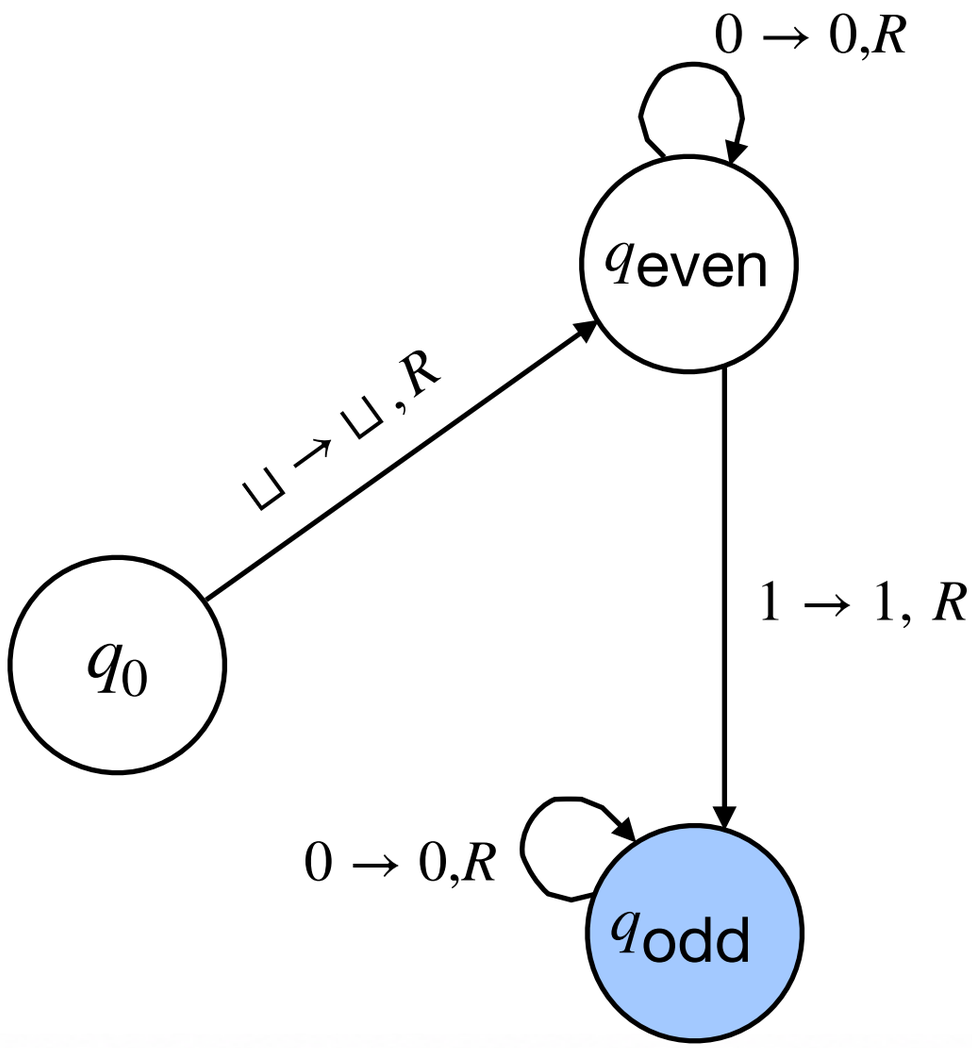
Example: Parity

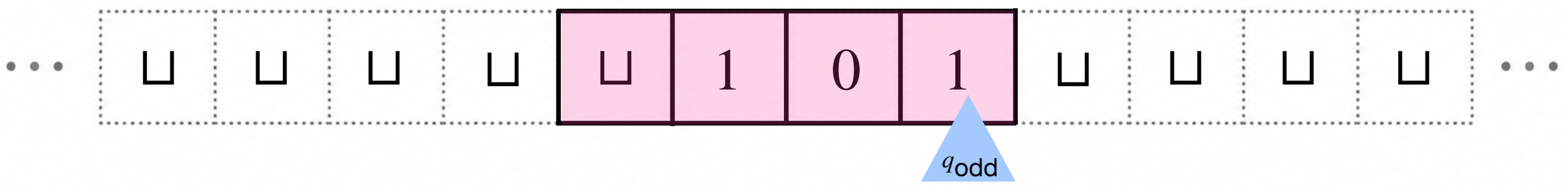


"odd # of 1's so far".

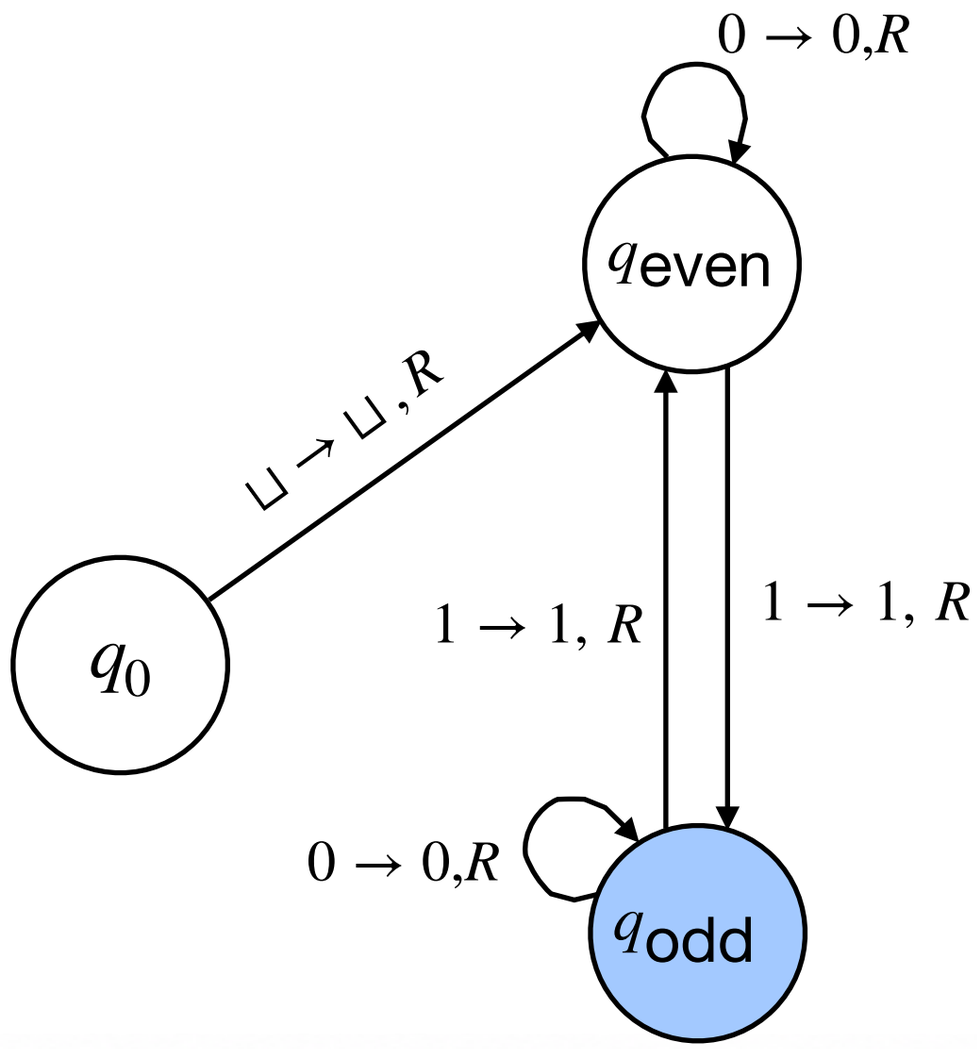


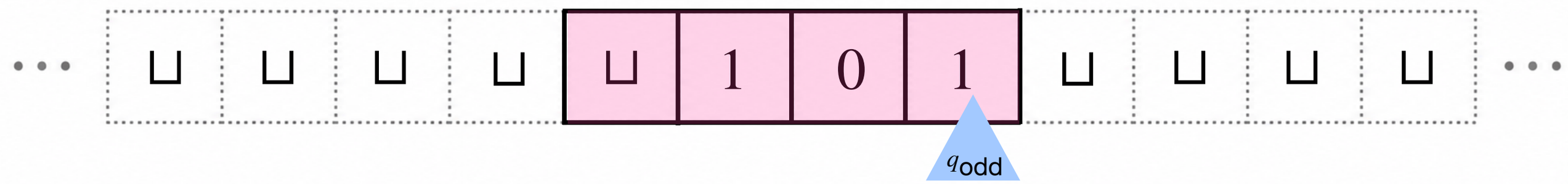
Example: Parity





Example: Parity





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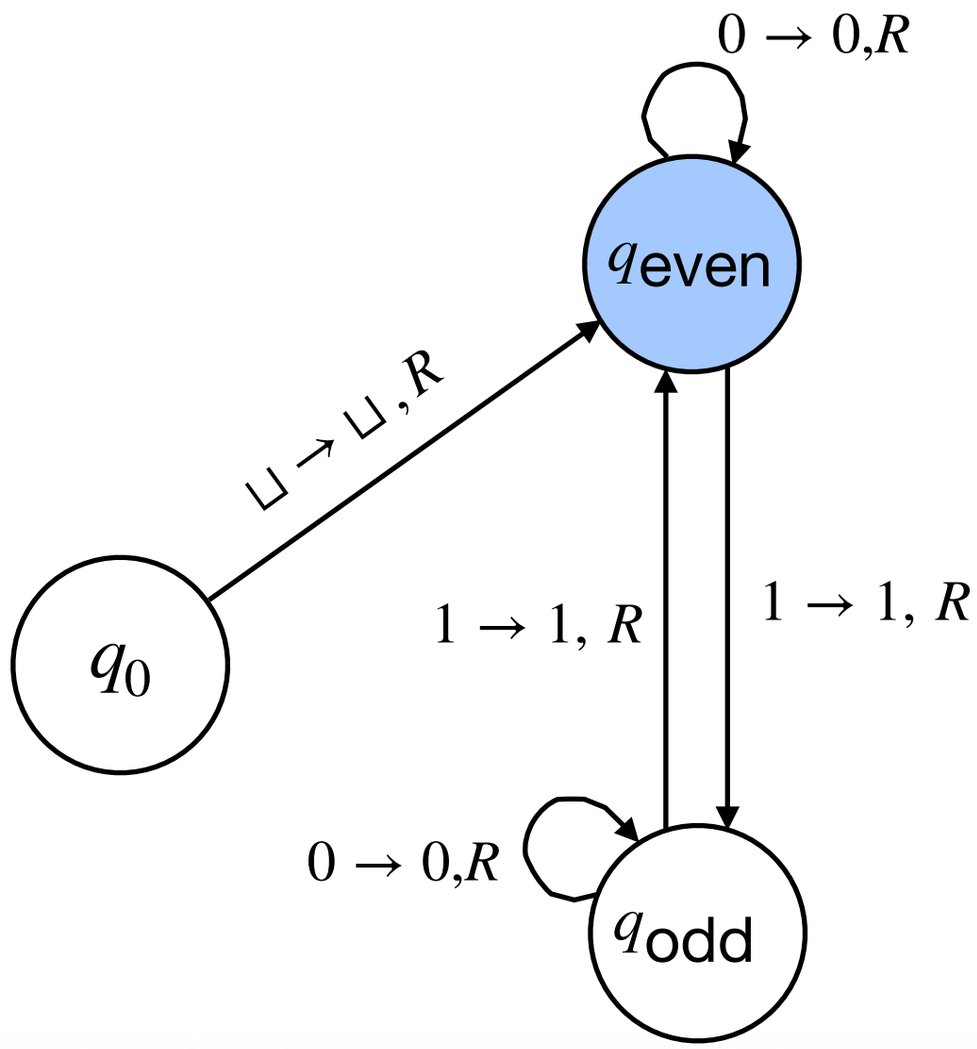

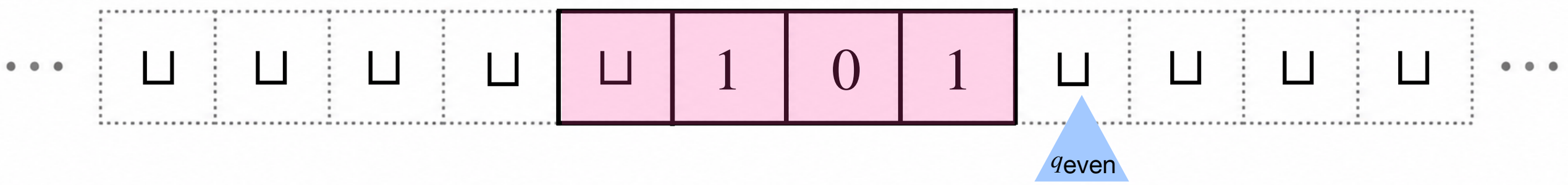

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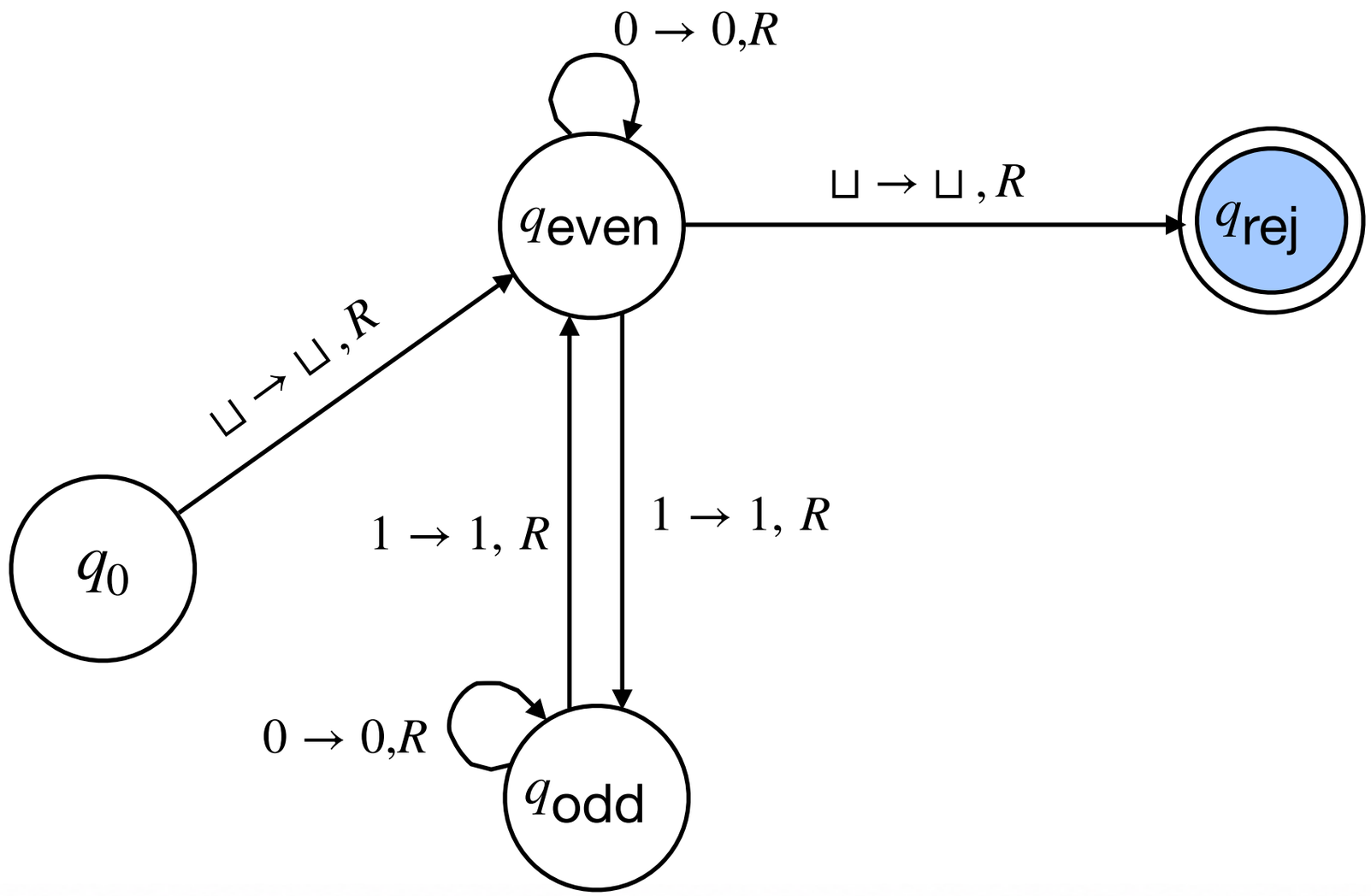

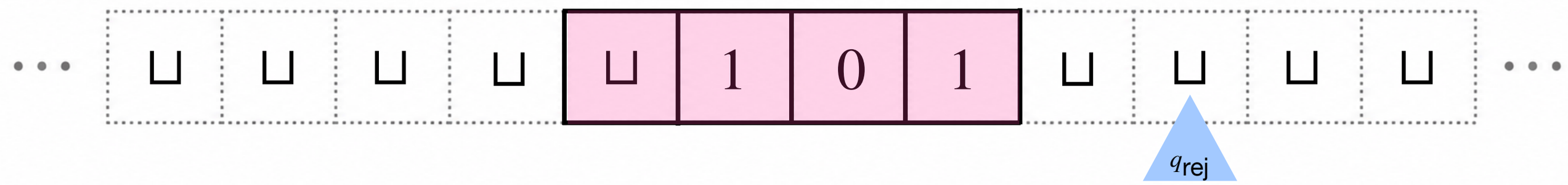

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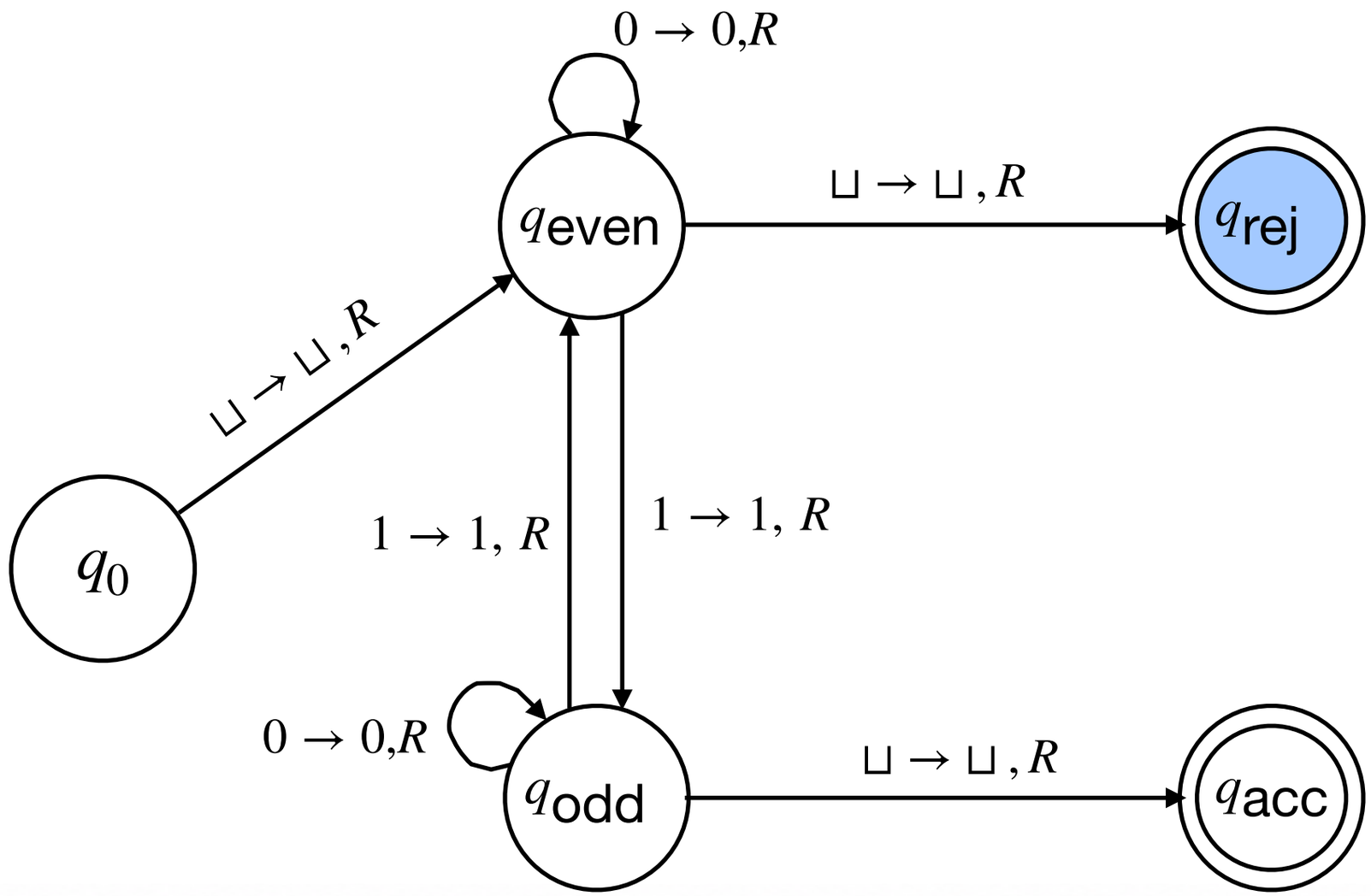
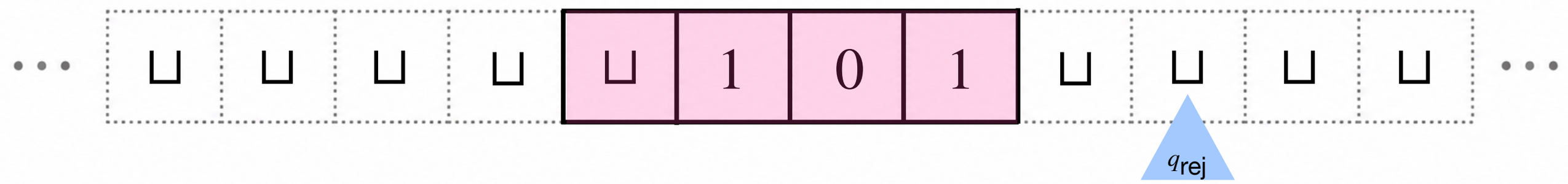
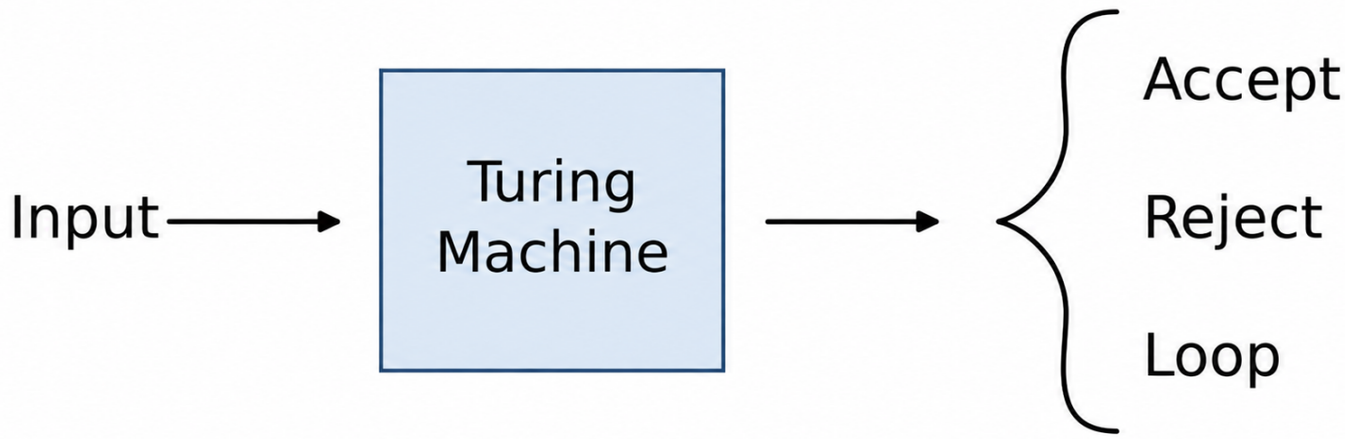




Decidable Languages

- A language L is decidable if and only if there exists a Turing Machine M that decides it.
- Otherwise, it is undecidable.

Recognizing a Language



- The Turing Machine M recognizes the language $L \subseteq \{0,1\}^*$ if for every $x \in \{0,1\}^*$, M accepts x if and only if $x \in L$.

- Note that by this definition, the machine M can reject or loop forever on inputs $x \notin L$.

Turing Machine as a Recognizer

- Every Turing Machine M recognizes some language $L(M) = \{x \in \{0,1\}^* \text{ such that } M \text{ accepts } x\}$.
- Not every Turing machine decides a language.

as it can loop forever.

Decidable and Recognizable Languages

- Decidable Languages $\subseteq$ Recognizable Languages
- Is this a proper subset?
- What is an example of an undecidable language?

Summary

- Turing Machines: a mathematical model of computation:
  - an unbounded tape,
  - a head that moves left/right, and reads/writes one cell,
  - has finitely many states
  - each decision is based on the state and the current symbol being read. Modeled as a transition function $\delta : Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$.
  - On input a Turing Machine may accept/reject/loop.

Summary

- M decides a language L if it accepts every $x \in L$ and rejects every $x \notin L$.

- M recognizes a language L if it accepts $x \iff x \in L$.

- A language L is decidable if and only if there exists a Turing machine that decides it. Otherwise it is undecidable.

- A language L is recognizable if and only if there exists a Turing machine that recognizes it.

Next Week: Formal Definition of TMs, and More Decidable Languages.