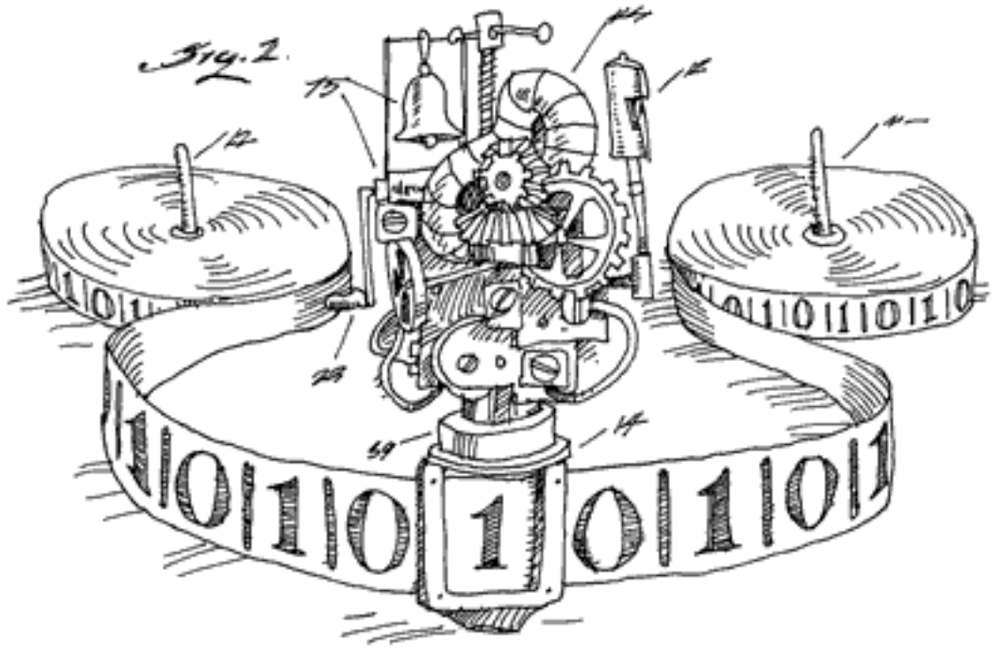
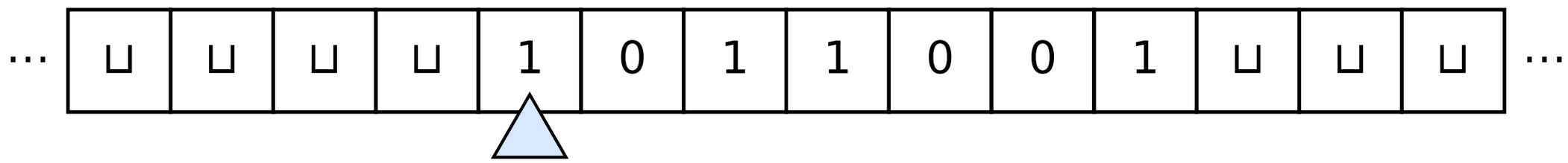
CSC4010: Introduction to Theoretical Computer Science



Lecture 4

Recap



Turing Machine Model:

- An infinite, one-dimensional “tape”. The tape divided into “cells”. Each cell contains a single symbol.
- There is a “head” pointing at one cell of the tape.
- The machine can be in one of finitely many “internal states”.
- In each step, the machine decides:
  - What to write in current cell,
  - Which cell to move in (left/right),
  - And, the next state.
- The decision is based only on the symbol being read and the current state the machine is in.

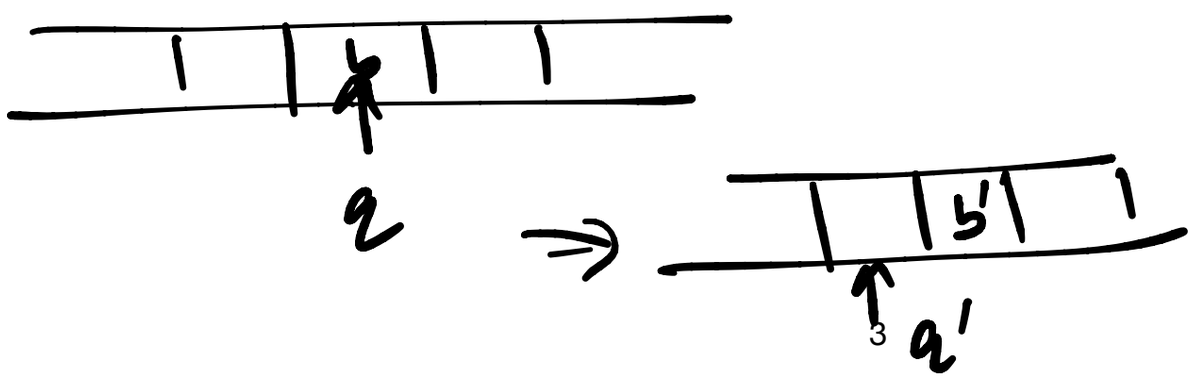
Formal Definition of Turing Machine

Mathematically, a Turing Machine is a 7-tuple
$M = (Q, q_{\text{accept}}, q_{\text{reject}}, q_0, \delta, \sqcup, \Sigma)$ such that:

- Q is a finite set (the set of "states".)
- $q_{\text{accept}}, q_{\text{reject}}, q_0 \in Q$ and $q_{\text{accept}} \neq q_{\text{reject}} \neq q_0$
- Σ is a finite set of symbols ("the tape alphabet").
- $\sqcup$ is a symbol (the "blank symbol").
- $\{0,1,\sqcup\} \subseteq \Sigma$ and $\sqcup \notin \{0,1\}$.
- δ is a function $\delta : Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$. (The "transition function".)

$\{0,1\}^* \mapsto$ Input.

$\delta(q,b) \mapsto (q', b', L)$



This definition is different from Sipser. (Both definitions are equivalent.)

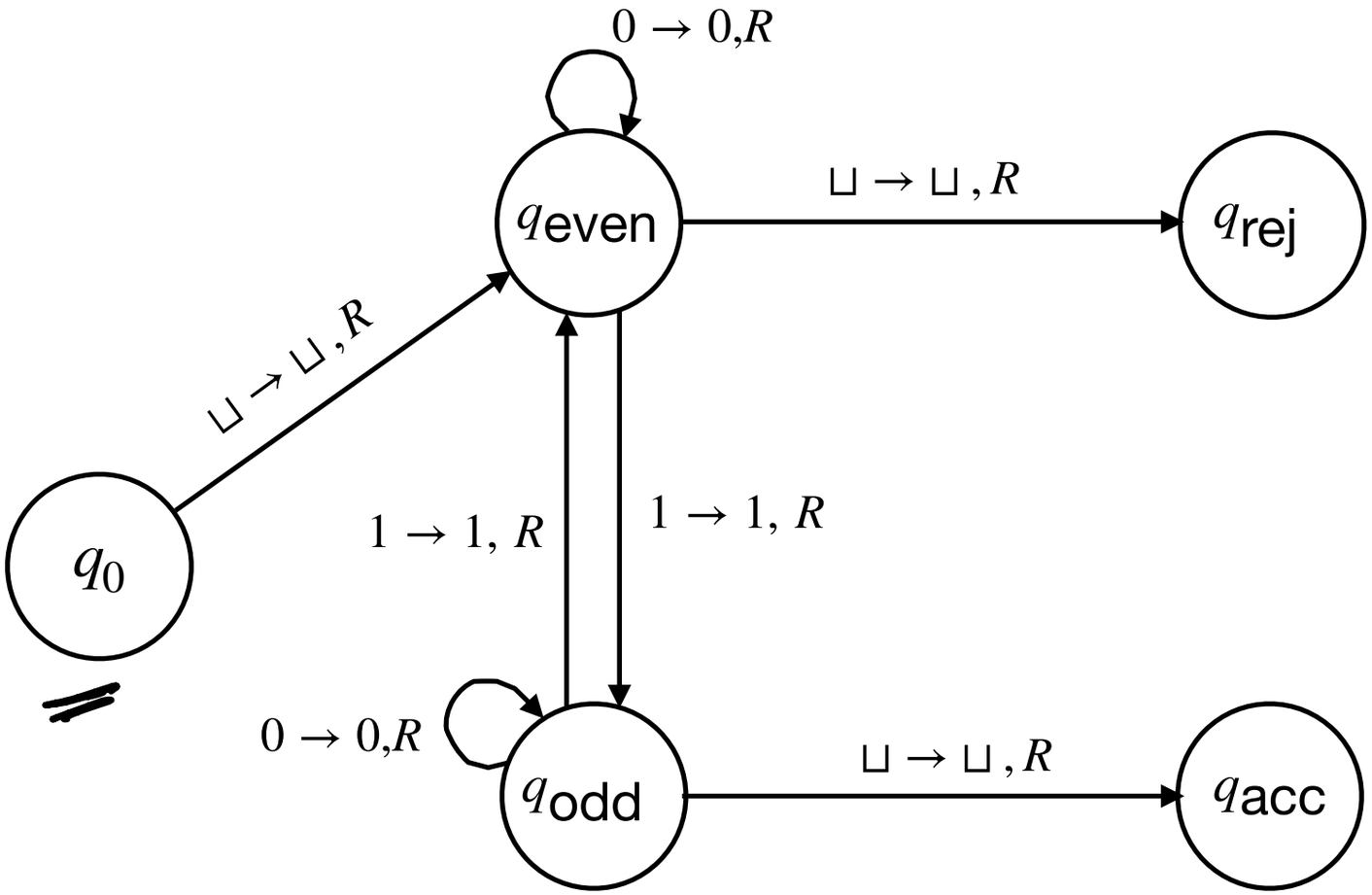
Example: Parity

- **Informal Problem Statement:** "Given $w \in \{0,1\}^*$, determine whether the number of ones in w is even or odd".
- **Can be formulated as:**

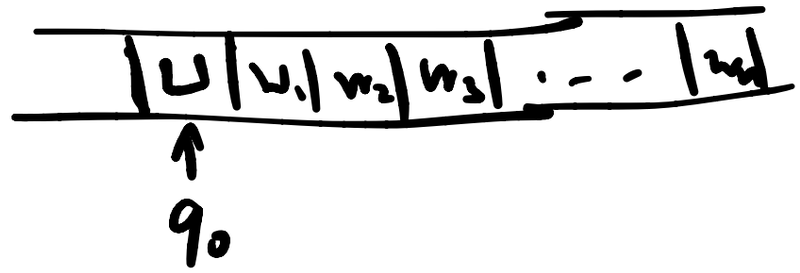
PARITY $= \{w \in \{0,1\}^*$ such that w has an odd number of 1's$\}$

- Design a Turing Machine for PARITY.

Example: Parity



Initialization



- $Q = \{q_0, q_{\text{acc}}, q_{\text{rej}}, q_{\text{odd}}, q_{\text{even}}\}$

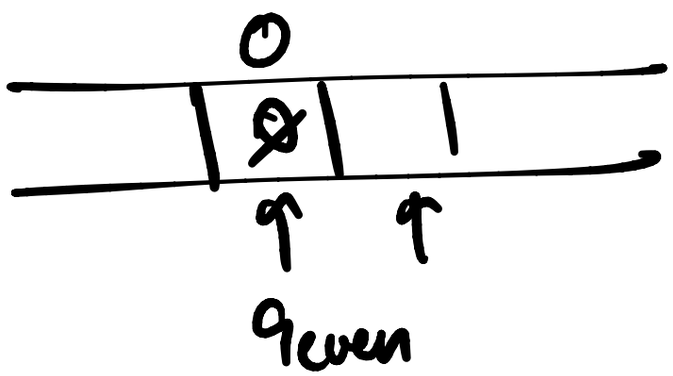
$\Sigma = \{0, 1, \sqcup\}$

- Let's look at δ :

	0	1	⊔
q_0			$(q_{even}, \sqcup, R)$
q_{even}	$(q_{even}, 0, R)$	$(q_{odd}, 1, R)$	
q_{odd}			
q_{accept}			
q_{reject}			

Q

$\delta : Q \times \Sigma \to Q \times \Sigma \times \{L, R\}$



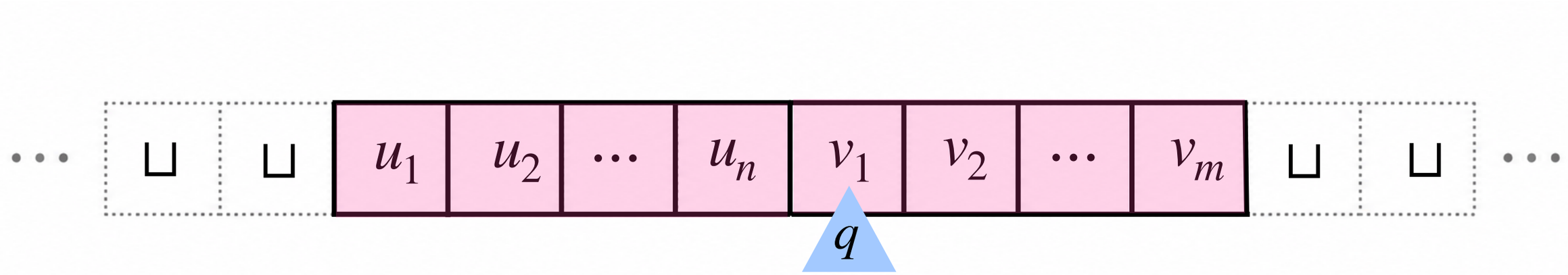
Transition Function

The transition function δ describes the local evolution of the computation.

What about the global evolution?

Configurations of a Turing Machine

- Let $M = (Q, q_{\text{accept}}, q_{\text{reject}}, q_0, \delta, \sqcup, \Sigma)$ be a Turing Machine.
- A configuration of M is a triple (u, q, v) where $u \in \Sigma^*, q \in Q, v \in \Sigma^*$ and $v \neq \epsilon$. Interpretation:
  - The tape currently contains uv .
  - The machine is in state q and the "head" is pointing at the first symbol of v .

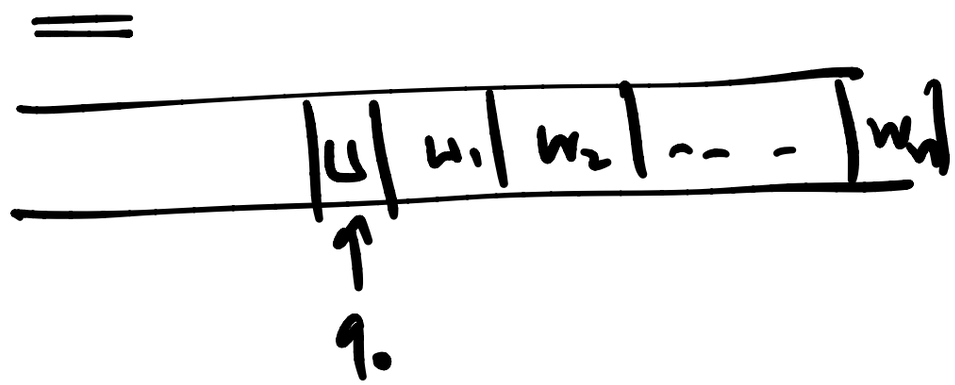


Configuration Shorthand

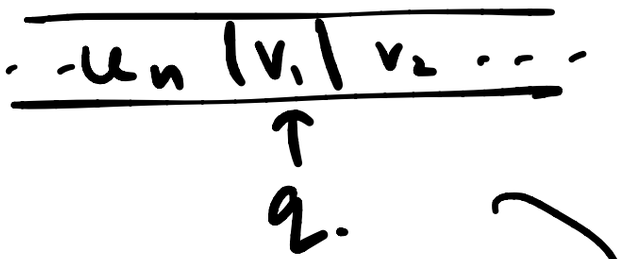
- Instead of (u, q, v), we often write uqv.
- The configuration uqv of a Turing Machine is a string over the alphabet $\Sigma \cup Q$.
- We can assume without loss of generality that $\Sigma \cap Q = \varnothing$.

The Initial Configuration

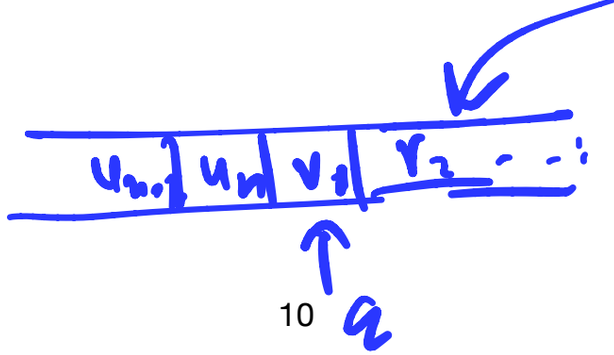
- Let $w \in \{0,1\}^*$ be the input.
- The initial configuration of M on w is $q_0 \sqcup w$.

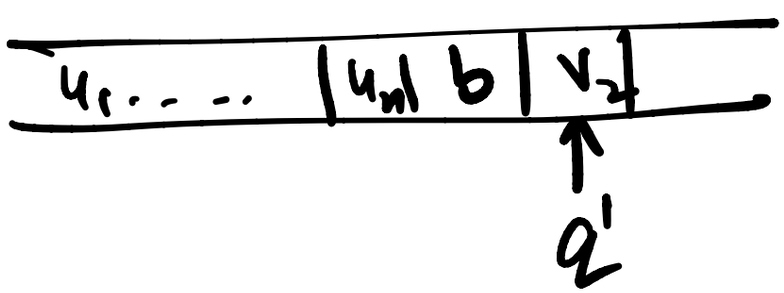


The "next" configuration

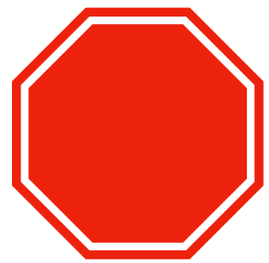


- For any configuration uqv, we define NEXT(uqv) as follows:
  - Break uqv into individual symbols: $uqv = u_1, u_2, \cdots, u_n q v_1, v_2, \cdots, v_m$.
  - If $\delta(q, v_1) = (q', b, R)$, then NEXT($uqv$) $= u_1 u_2 \cdots u_n b q' v_2, v_3 \cdots v_m$.
    - **Edge Case:** If $m = 1$, then NEXT(uqv) $= u_1 u_2 \cdots u_n b q' \sqcup$.
  - If $\delta(q, v_1) = (q', b, L)$, then NEXT($uqv$) $= u_1 u_2 \cdots u_{n-1} q' u_n b v_2, v_3 \cdots v_m$.
    - **Edge Case:** If $n = 0$, then NEXT(uqv) $= q' \sqcup b v_2 v_3 \cdots v_m$.
  - We write NEXT$_M(uqv)$ if M is not clear from context.

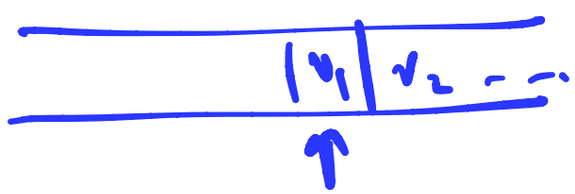




Halting Configurations



- An accepting configuration is of the form $uq_{\text{accept}}v$.
- An rejecting configuration is of the form $uq_{\text{reject}}v$.



- A halting configuration is either an accepting or a rejecting configuration.

Computational History

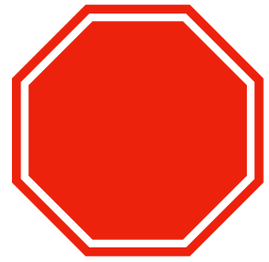
max $i \in [T]$ $|C_i|$ = space complexity.

- Let $w \in \{0,1\}^*$ be an input string.
- Let C_0 be the initial configuration of M on w i.e. $C_0 = q_0 \sqcup w$.
- Inductively, for each $i \in \mathbb{N}$, let NEXT$(C_i) = C_{i+1}$.
- The computational history of M on w is the sequence $C_1, C_2, \cdots, C_T$, where C_T is the first halting configuration in the sequence.

T = time complexity

- If there is no such C_T, then the computational history is infinite: $C_1, C_2, C_3, \cdots$.

Halting and Looping



- If the computational history of M on w is finite, we say M halts on w .
- Otherwise, M loops on w .

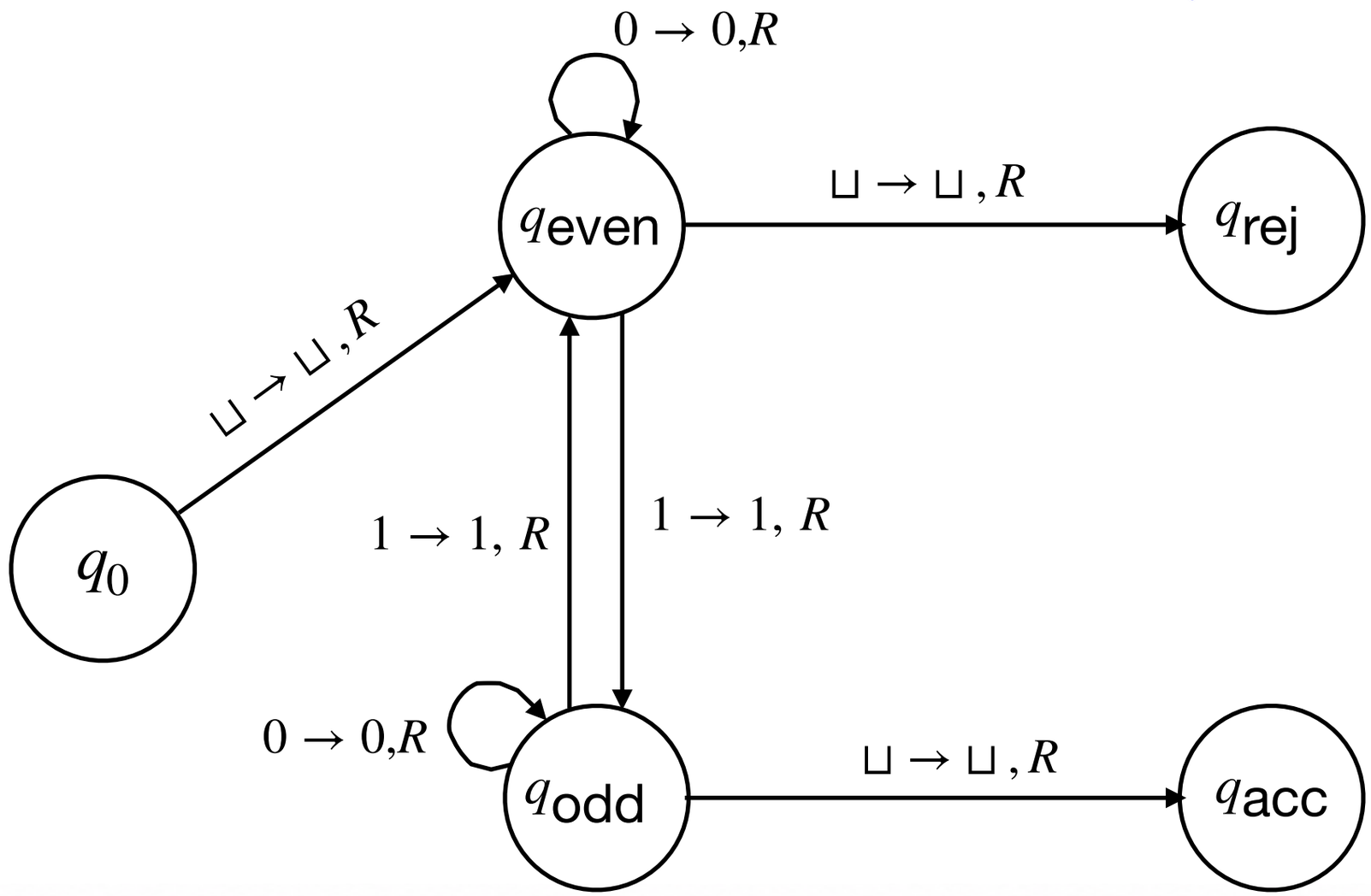
Accepting and Rejecting

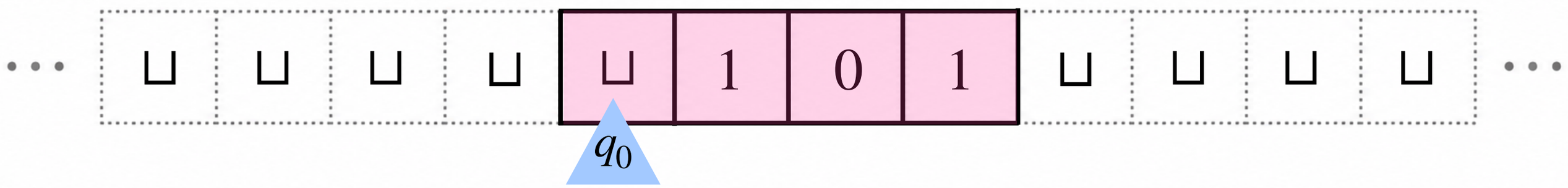
- Suppose M halts on w .
- The computational history is finite, $C_0, C_1, \cdots, C_T$.
- If C_T is a accepting configuration, we say M accepts w .
- If C_T is a rejecting configuration, we say M rejects w .

Example: Parity

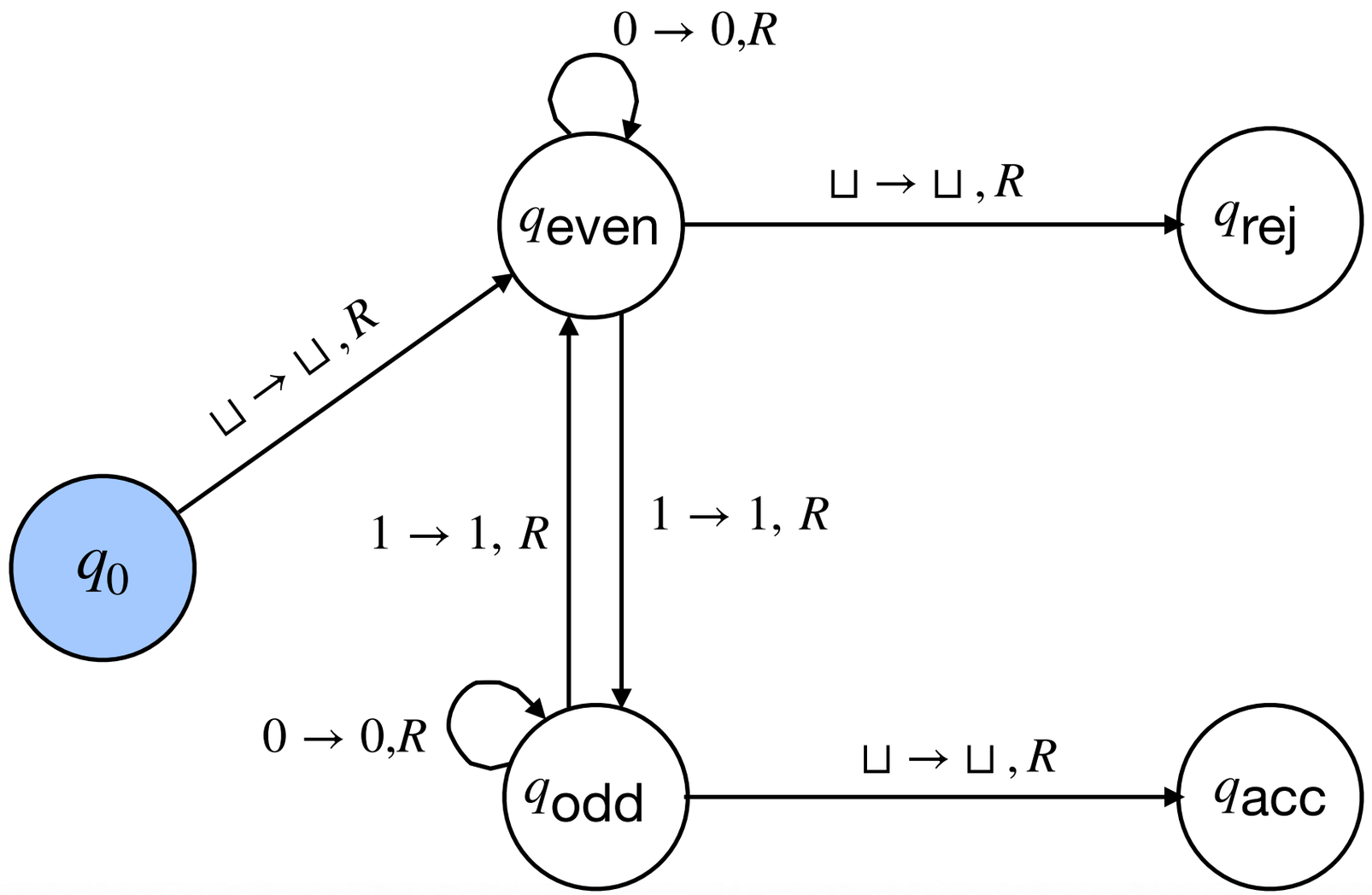
$q_0 \sqcup 101$

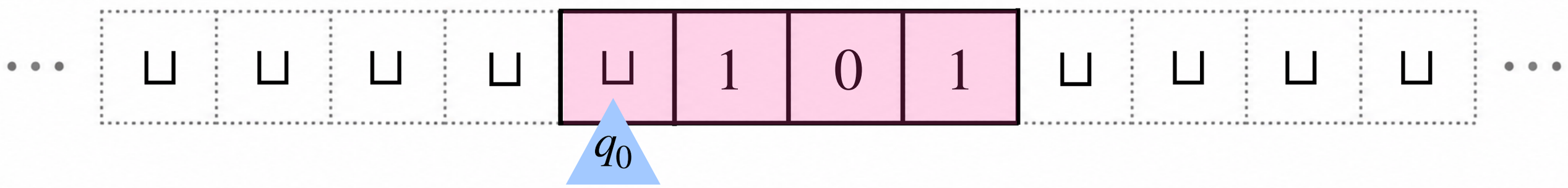
$\sqcup q_0 101$



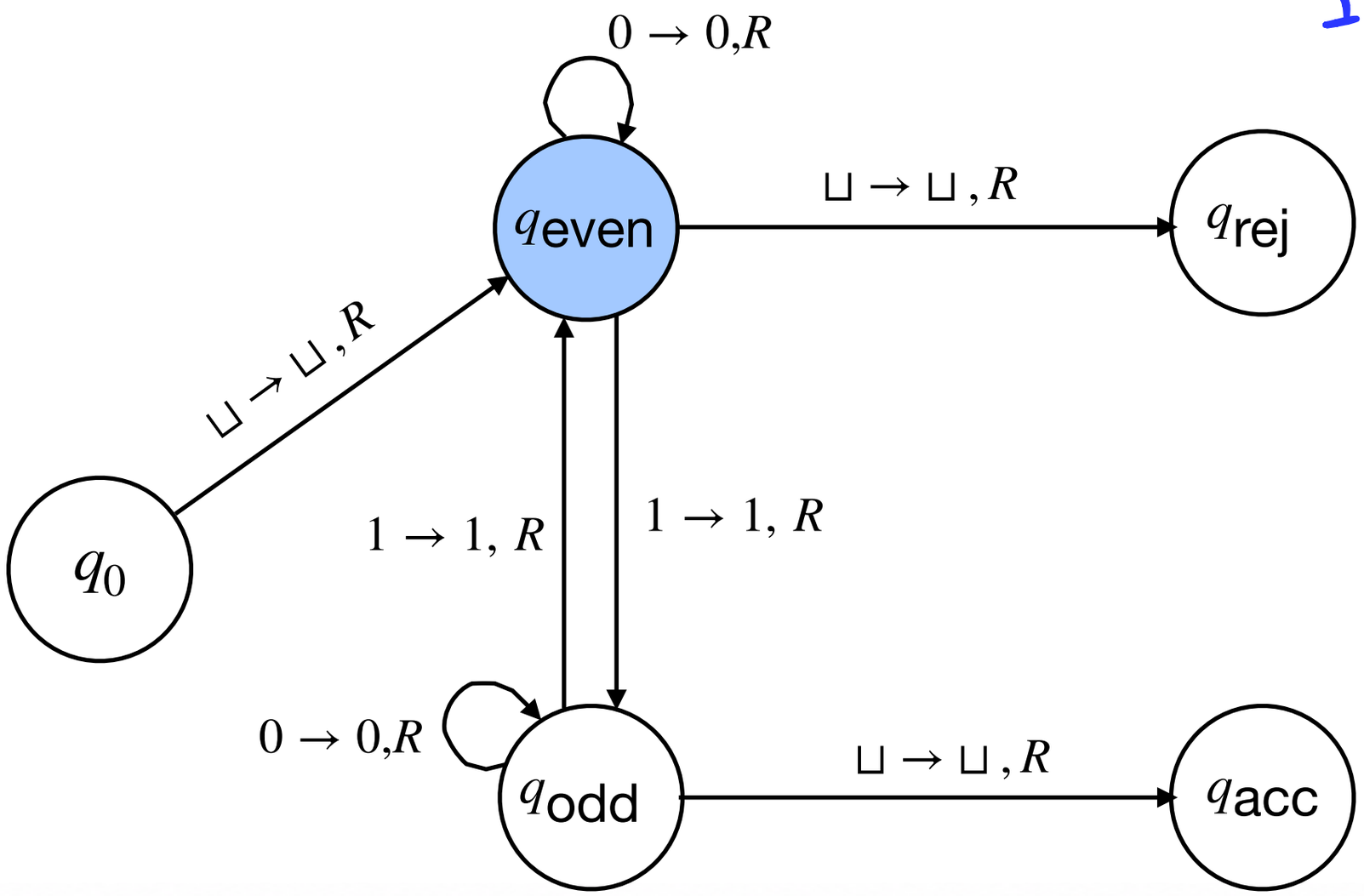


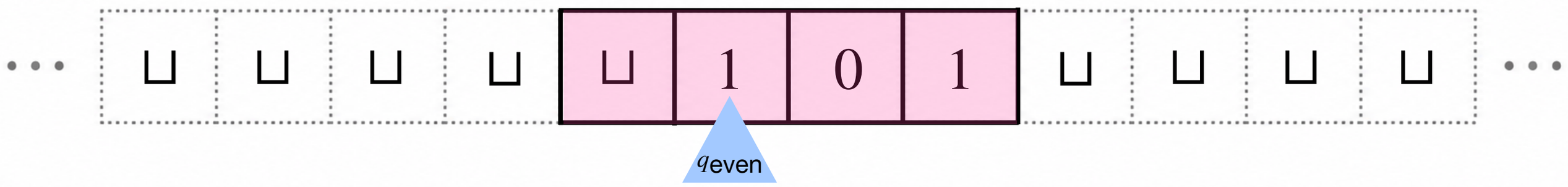
Example: Parity



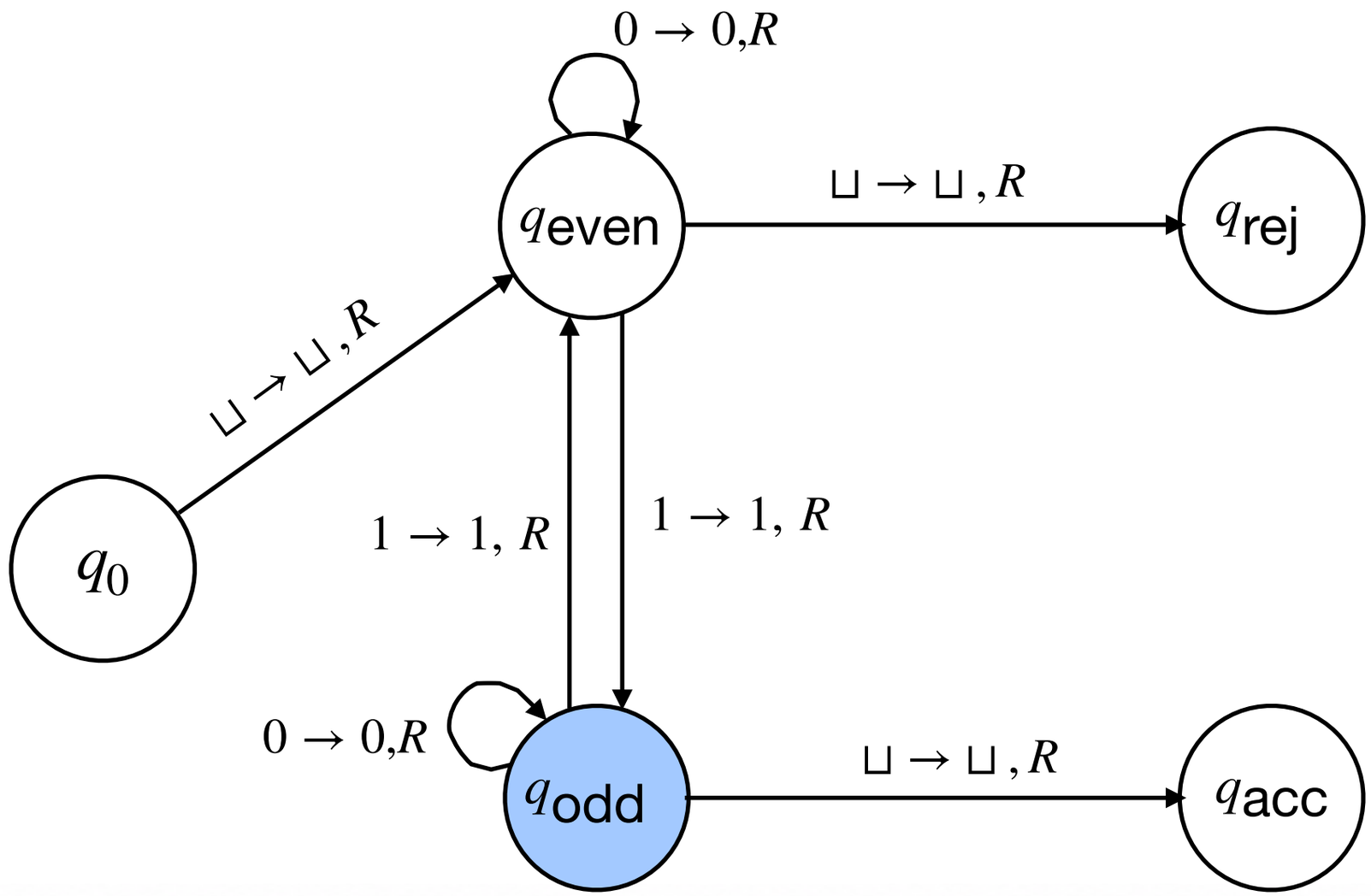


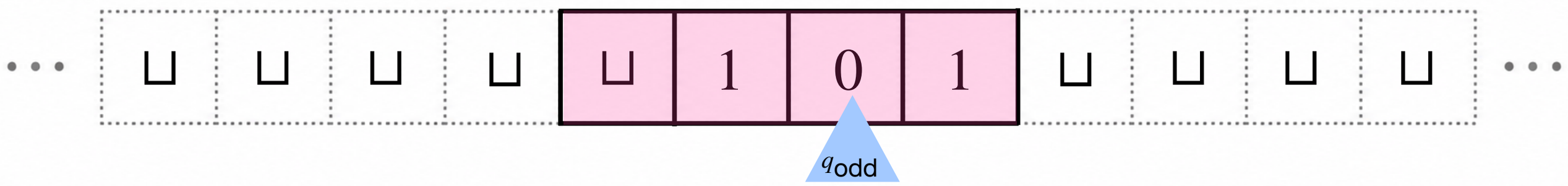
Example: Parity



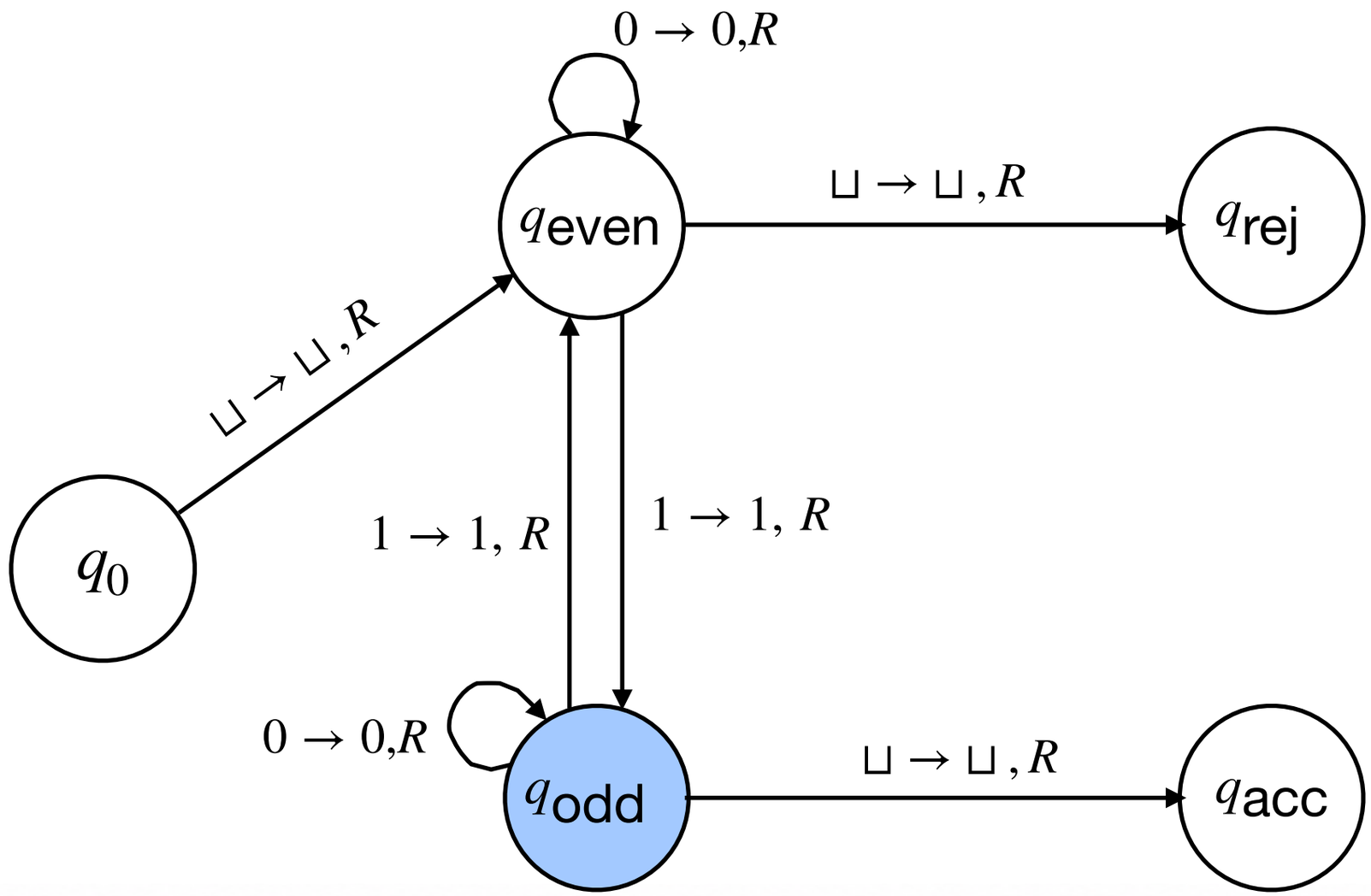


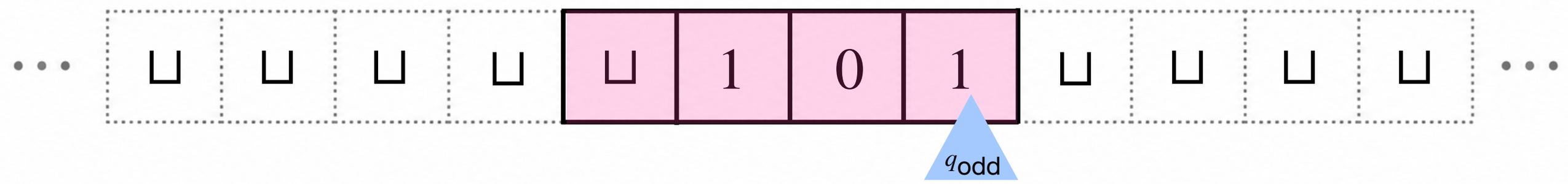
Example: Parity



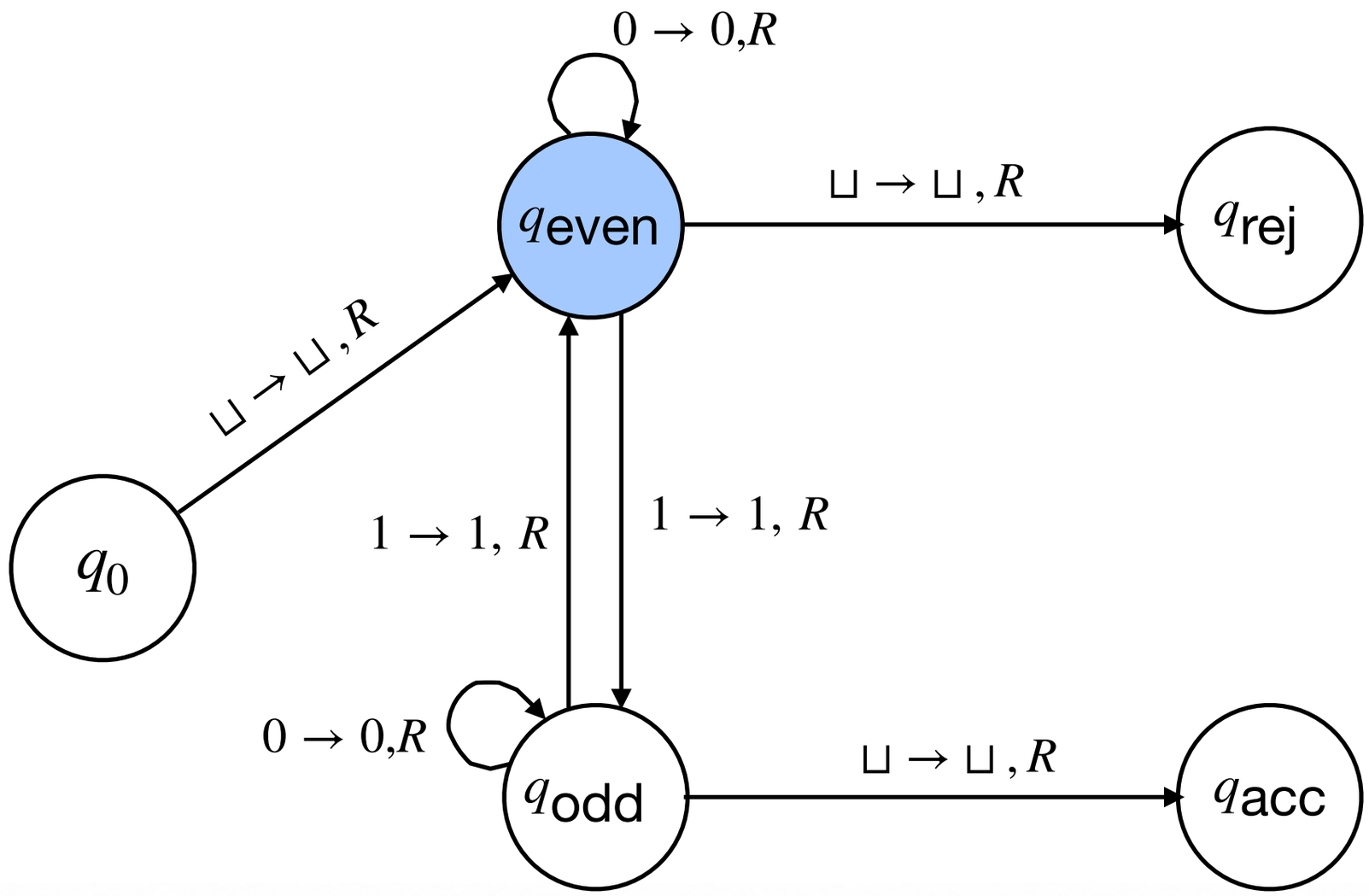


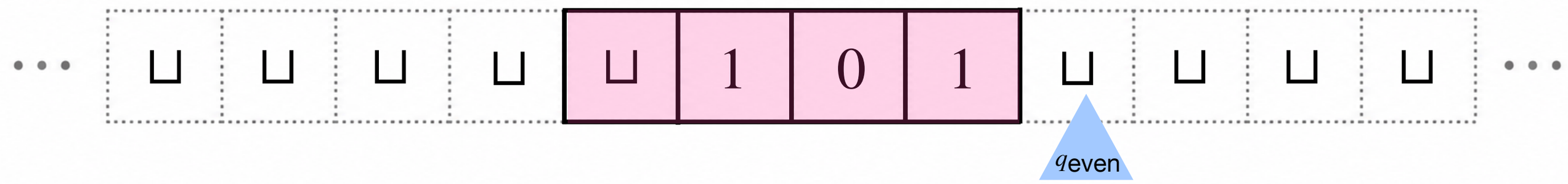
Example: Parity



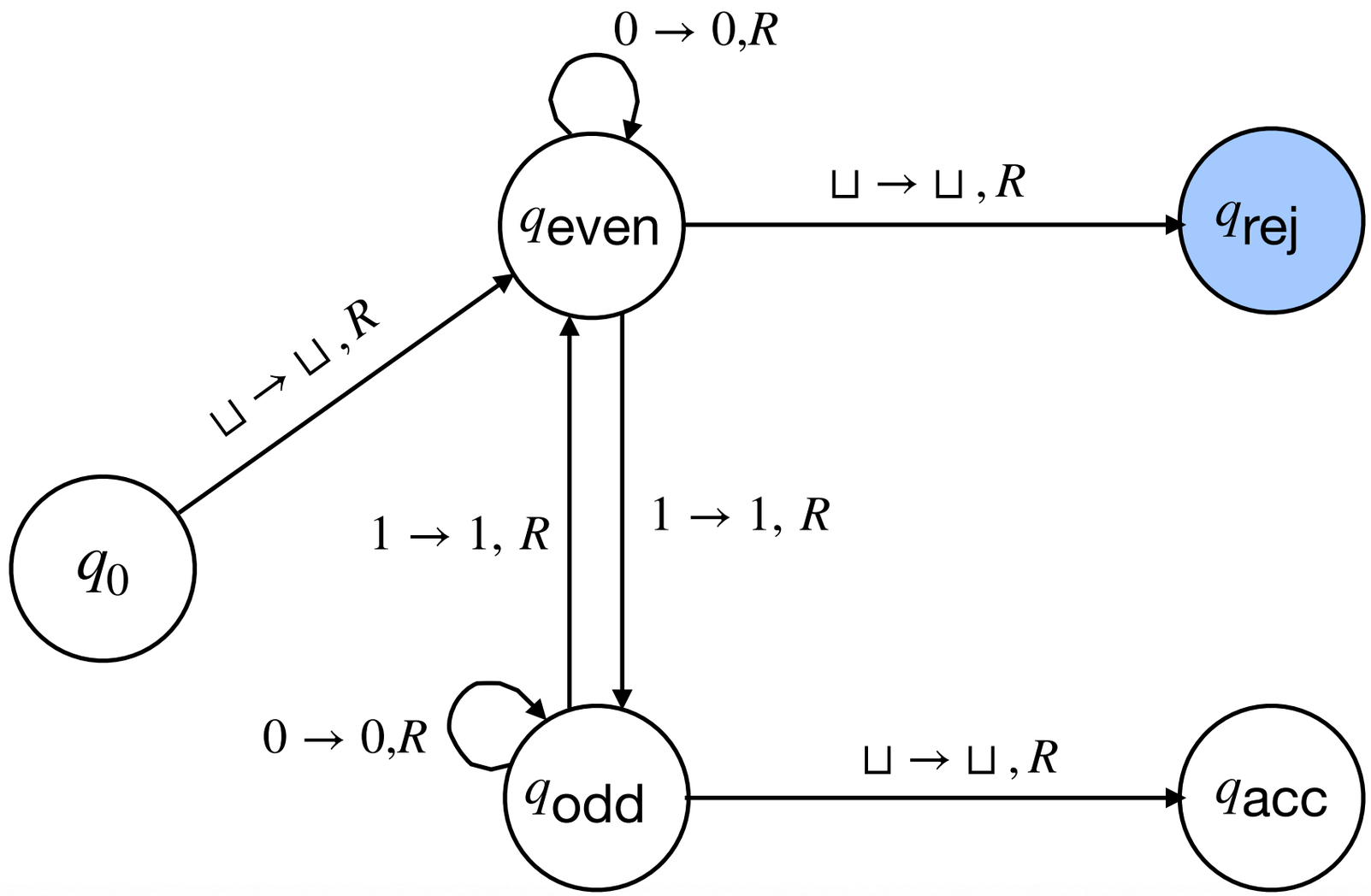


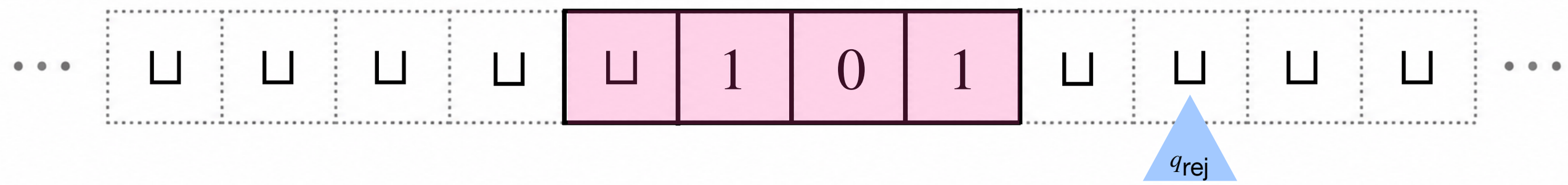
Example: Parity





Example: Parity





Which problems can be solved through computation?

Deciding a Language

- Let M be a Turing Machine and let $L \subseteq \{0,1\}^*$.
- We say that M decides L if
  - M accepts every $x \in L$.
  - M rejects every $x \in \{0,1\}^* \setminus L$.

◉ This is mathematical model of what it means to "solve a problem".

Examples of Decidable Languages

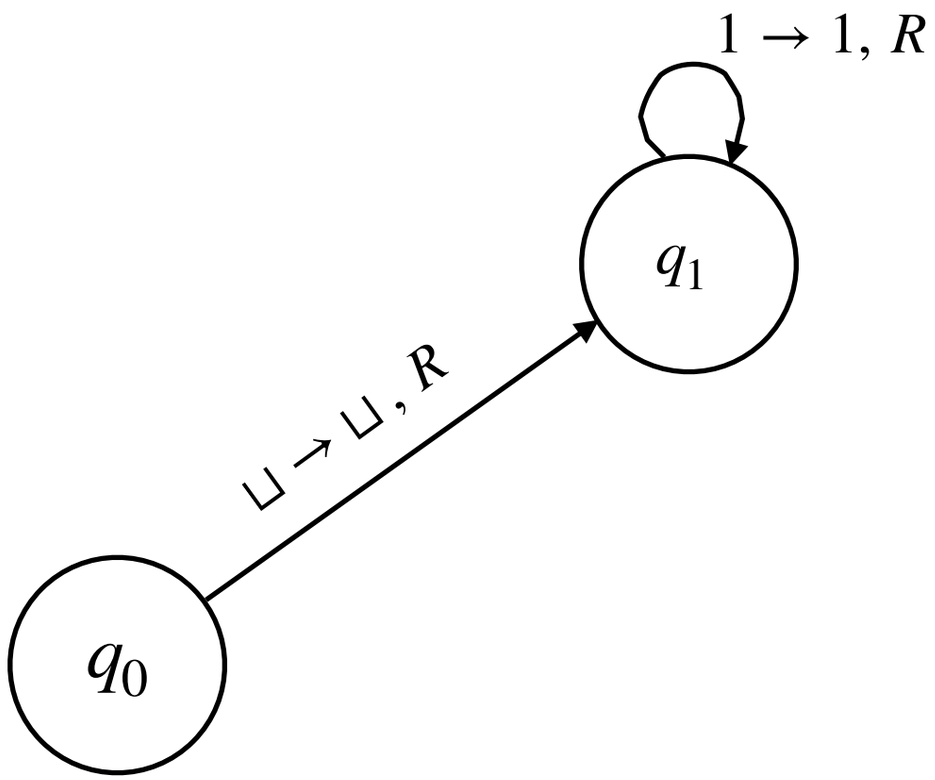
Example: Strings containing 01

- **Informal Problem Statement:** "Given $x \in \{0,1\}^*$, determine whether it contains 01 as a substring".
- **Can be formulated as:**

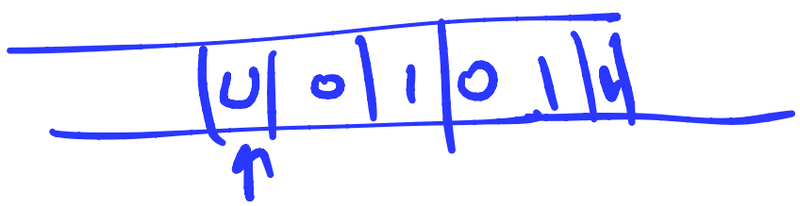
$$L_{01} = \{w \in \{0,1\}^* \text{ such that } w \text{ contains substring } 01\}.$$

- Design a Turing Machine for L_{01}.

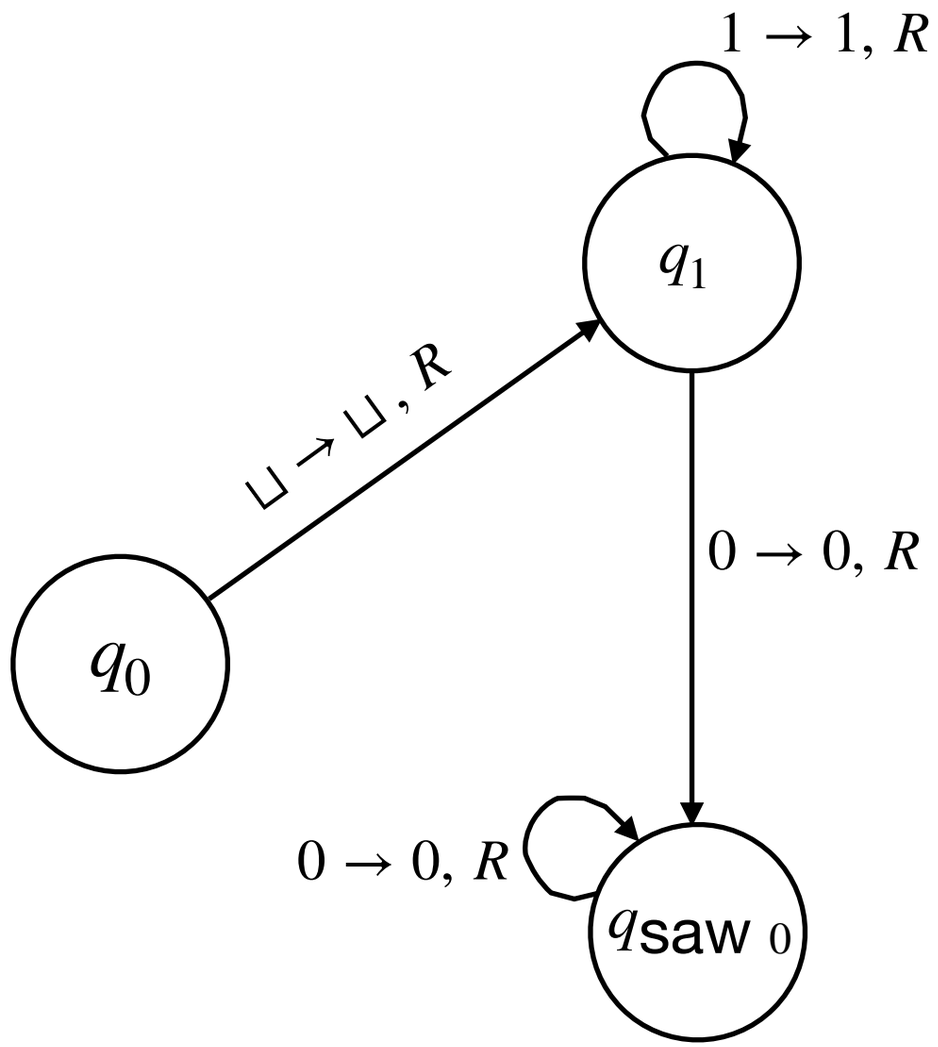
Example: L_{01}



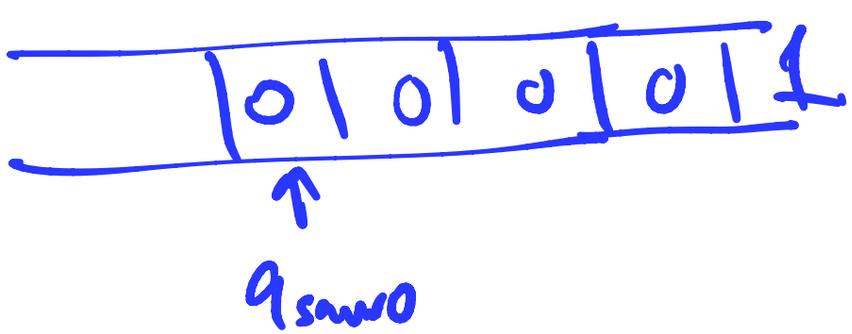
- $q_0 =$ "Start State"
- $q_1 =$ "Haven't seen a 0 yet"



Example: L_{01}

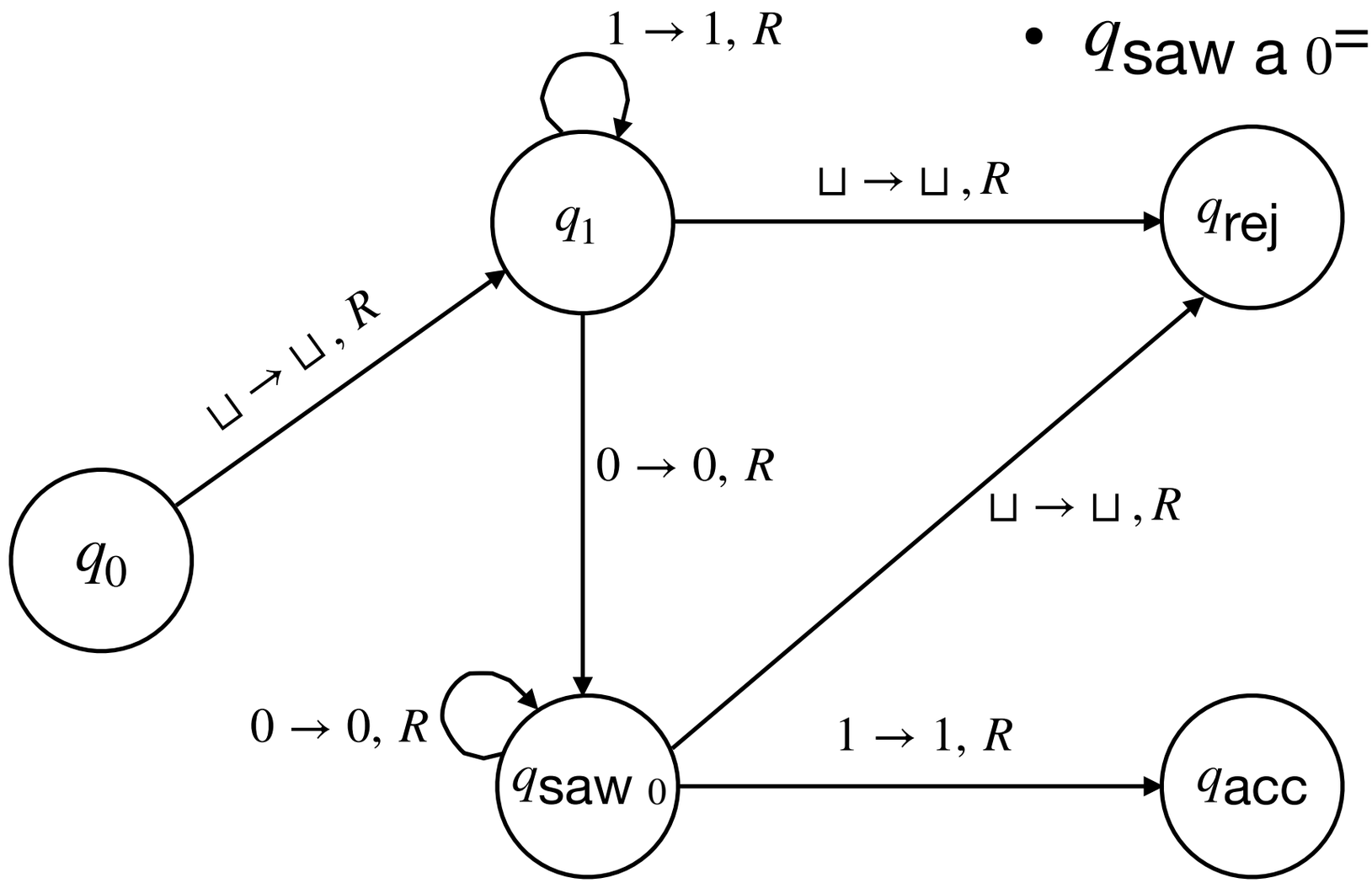


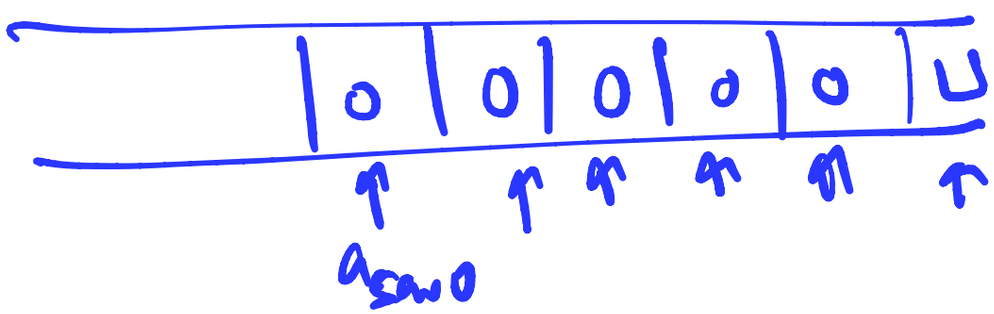
- q_0 = "Start State"
- q_1 = "Haven't seen a 0 yet"
- $q_{\text{saw a } 0}$="saw a 0"

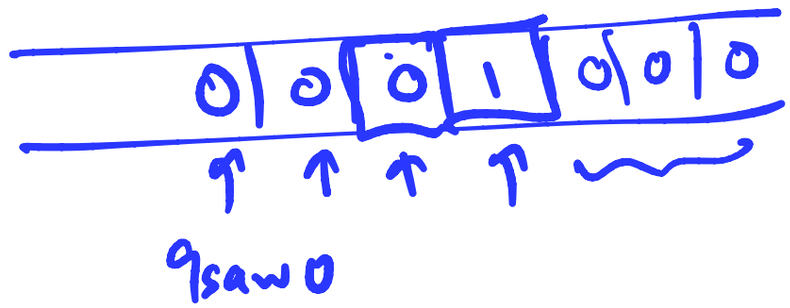


Example: L_{01}

- q_0 = "Start State"
- q_1 = "Haven't seen a 0 yet"
- $q_{\text{saw a } 0}$="saw a 0"





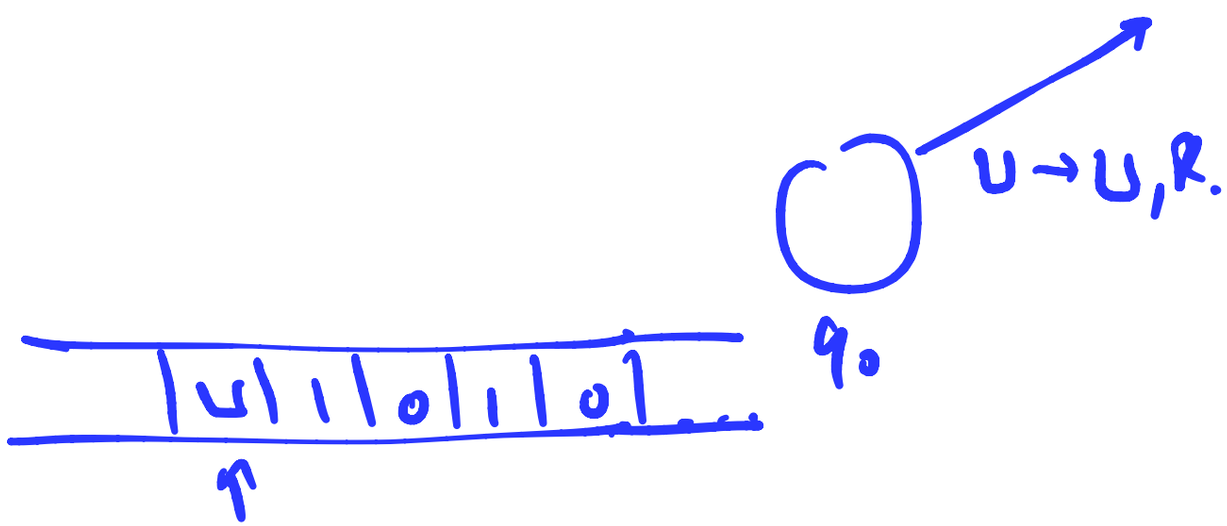


Example: Strings Ending in 1

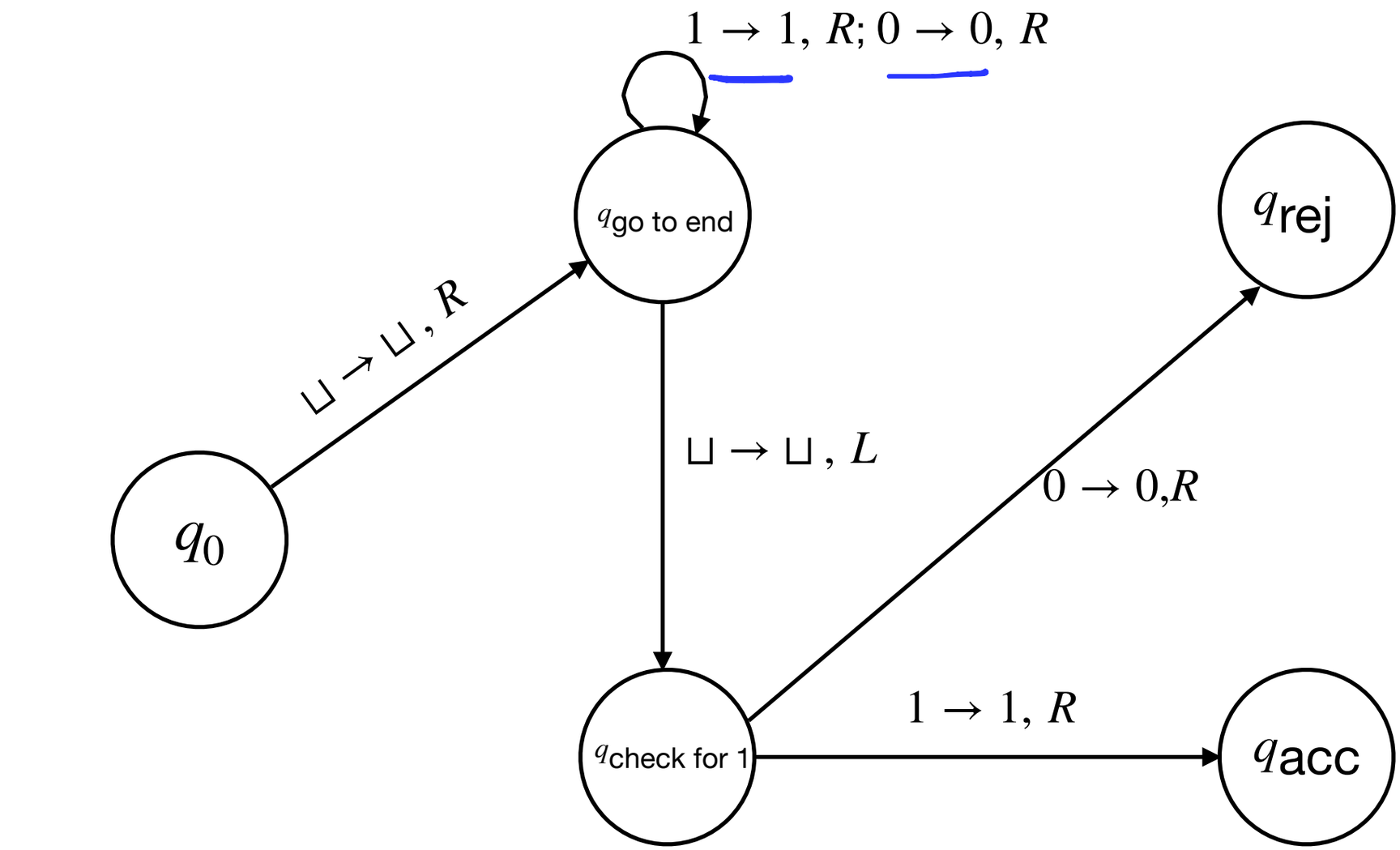
- **Informal Problem Statement:** "Given $x \in \{0,1\}^*$, determine whether it ends in 1".
- **Can be formulated as:**

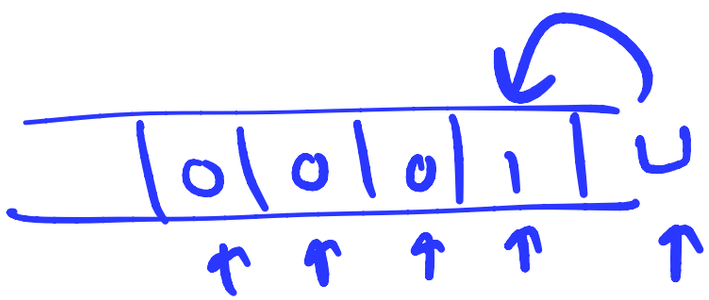
$$L_1 = \{w \in \{0,1\}^* \text{ such that } w \text{ ends in } 1\}.$$

- Design a Turing Machine for L_1.



Example: L_1

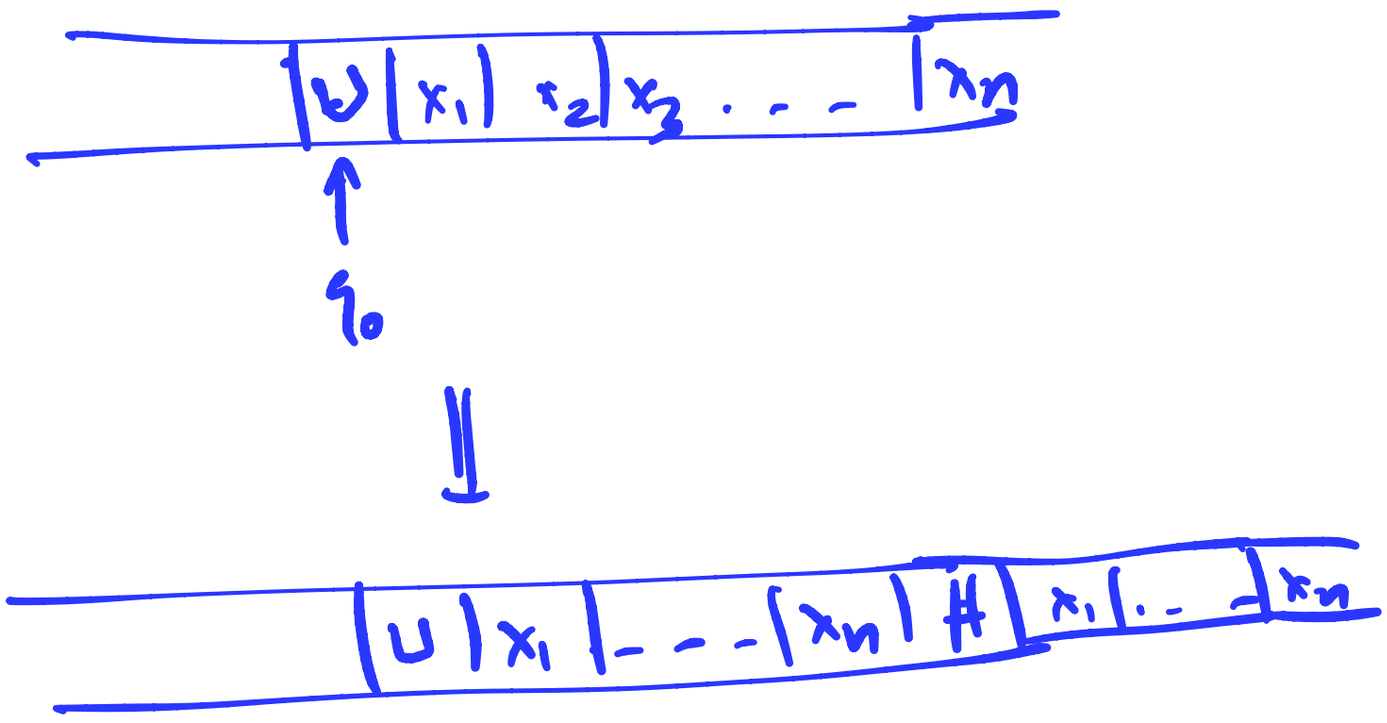

$1 \to 1,\ R;\ 0 \to 0,\ R$
$q_{\text{go to end}}$
q_{rej}
$\sqcup \to \sqcup,\ R$
$\sqcup \to \sqcup,\ L$
$0 \to 0,R$
q_0
$q_{\text{check for 1}}$
$1 \to 1,\ R$
q_{acc}


0 0 0 1 ⊔

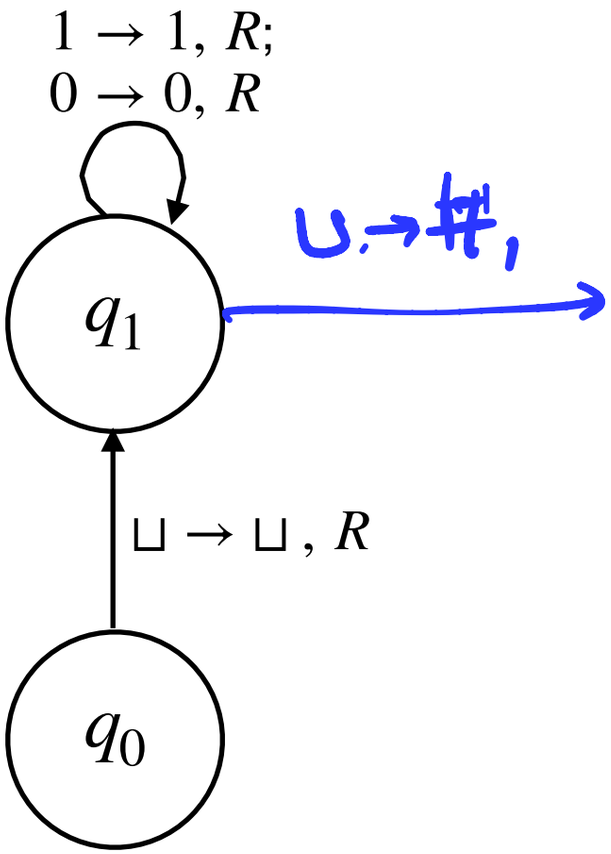
More Examples

Example: TM for COPY

- Takes as input a string $x \in \{0,1\}^*$ and replaces it with $x\#x$ on the tape and accepts.

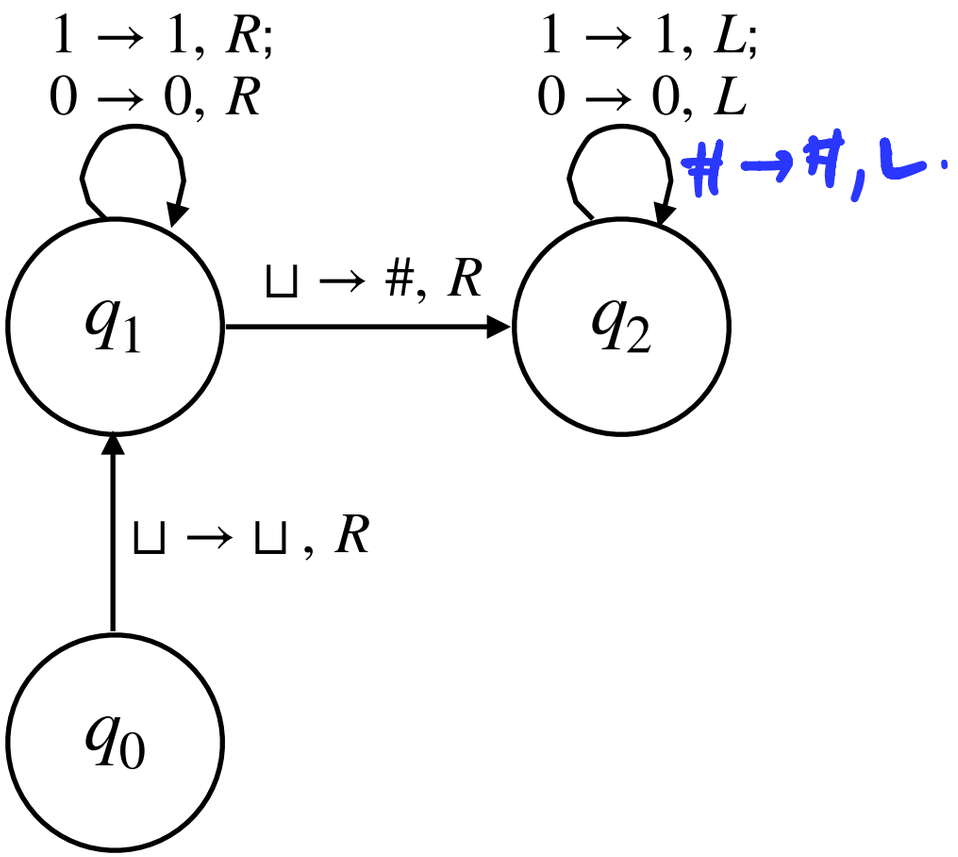


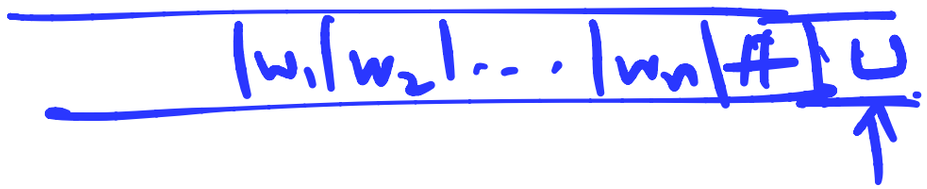
Example: TM for COPY

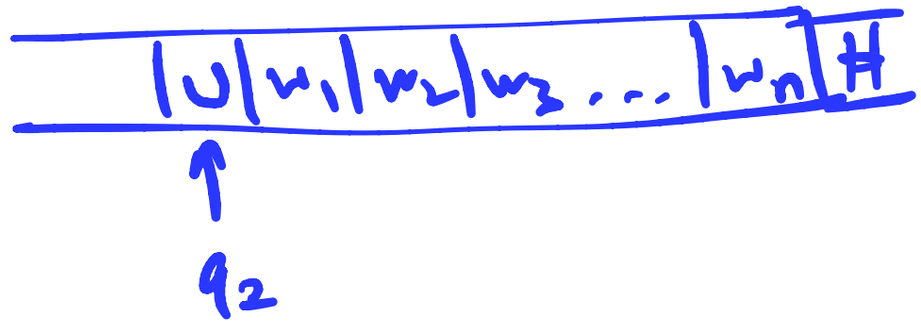


- q_0 = "Start State"
- q_1 = ``Append # to end of w”

Example: TM for COPY

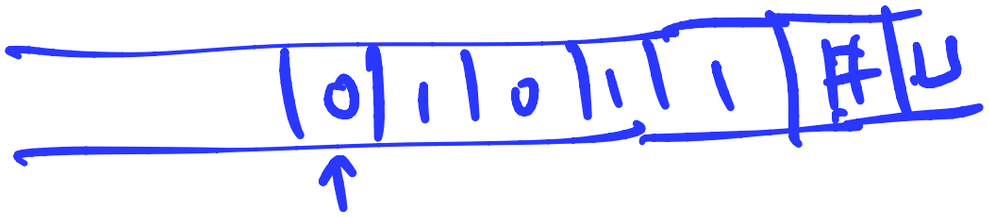


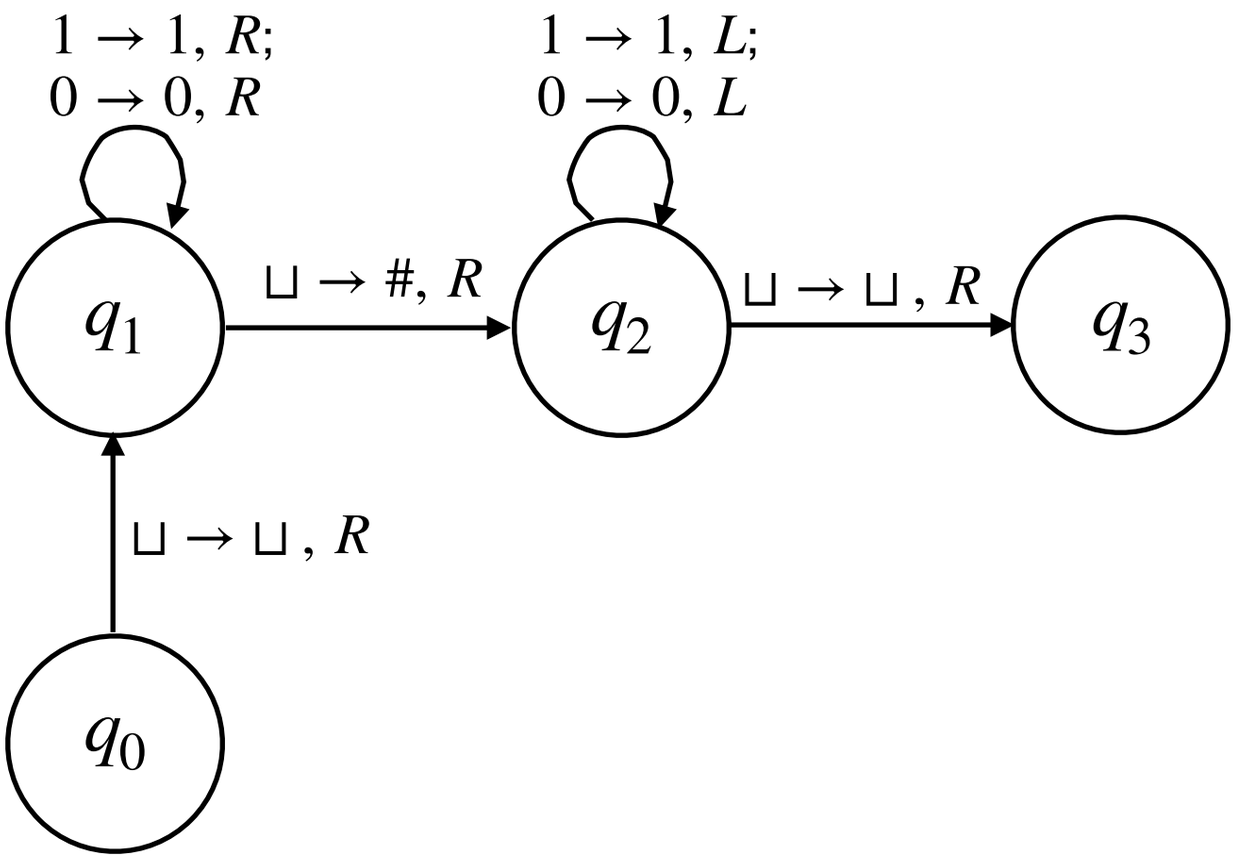




- q_0 = "Start State"
- q_1 = ``Append # to end of w”
- q_2 = ``Goto beginning of string”

Example: TM for COPY





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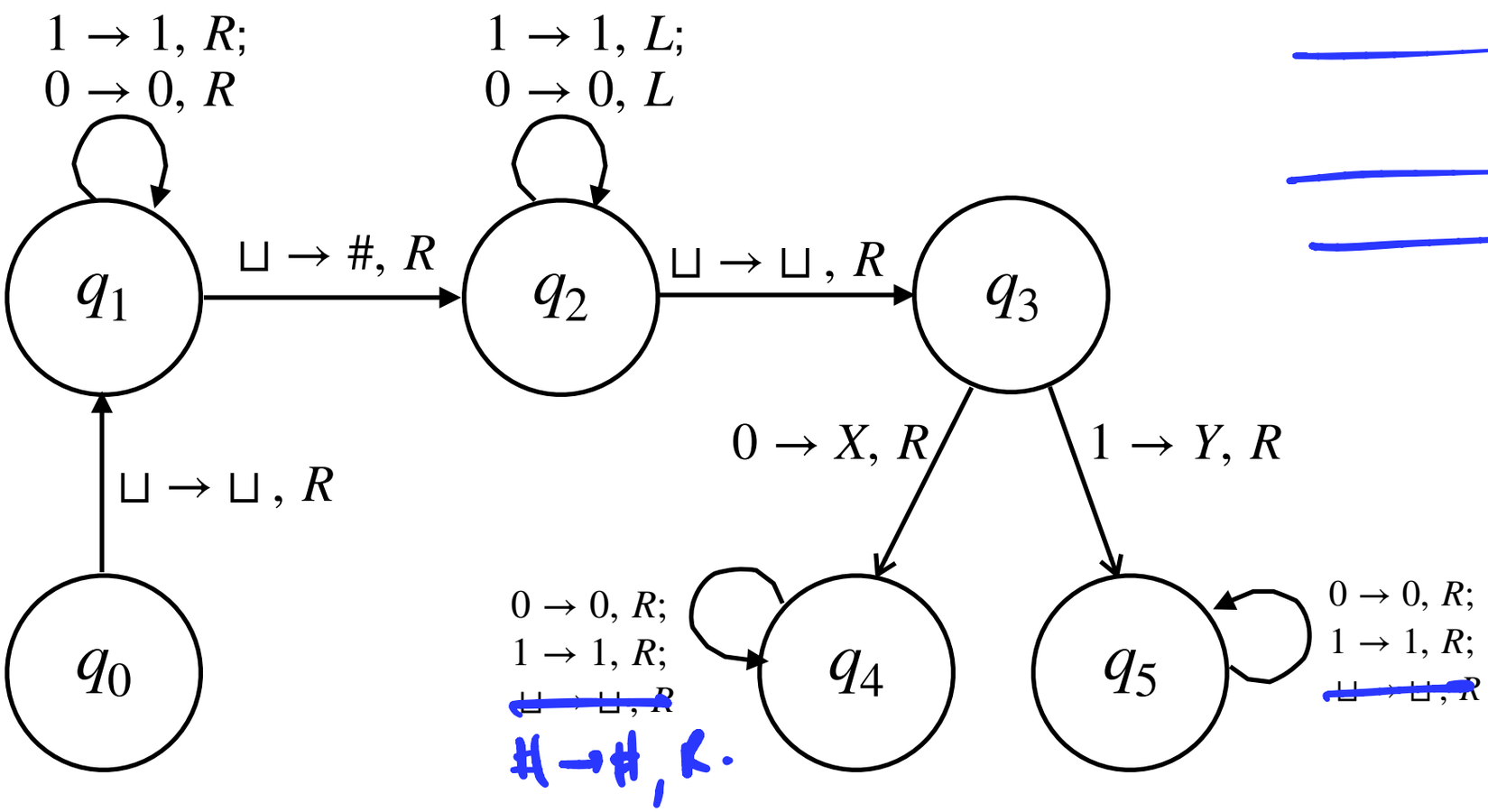

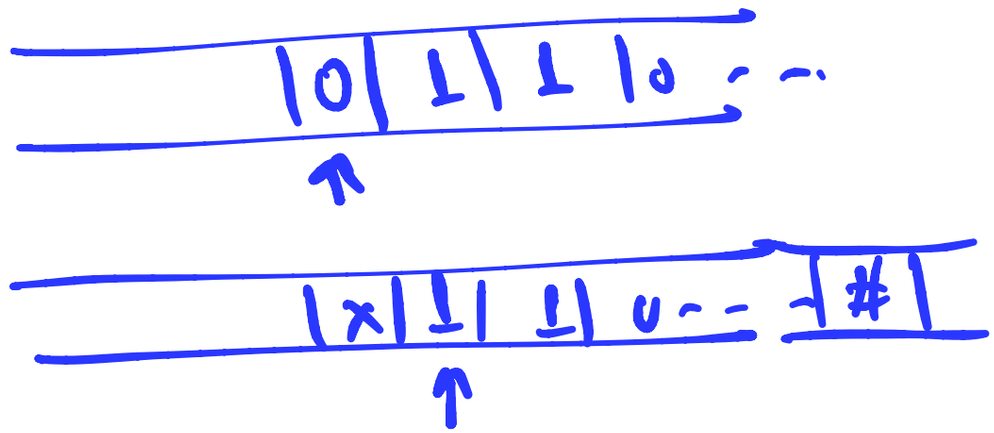

- q_0 = "Start State"
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- q_5 = "read a 1"

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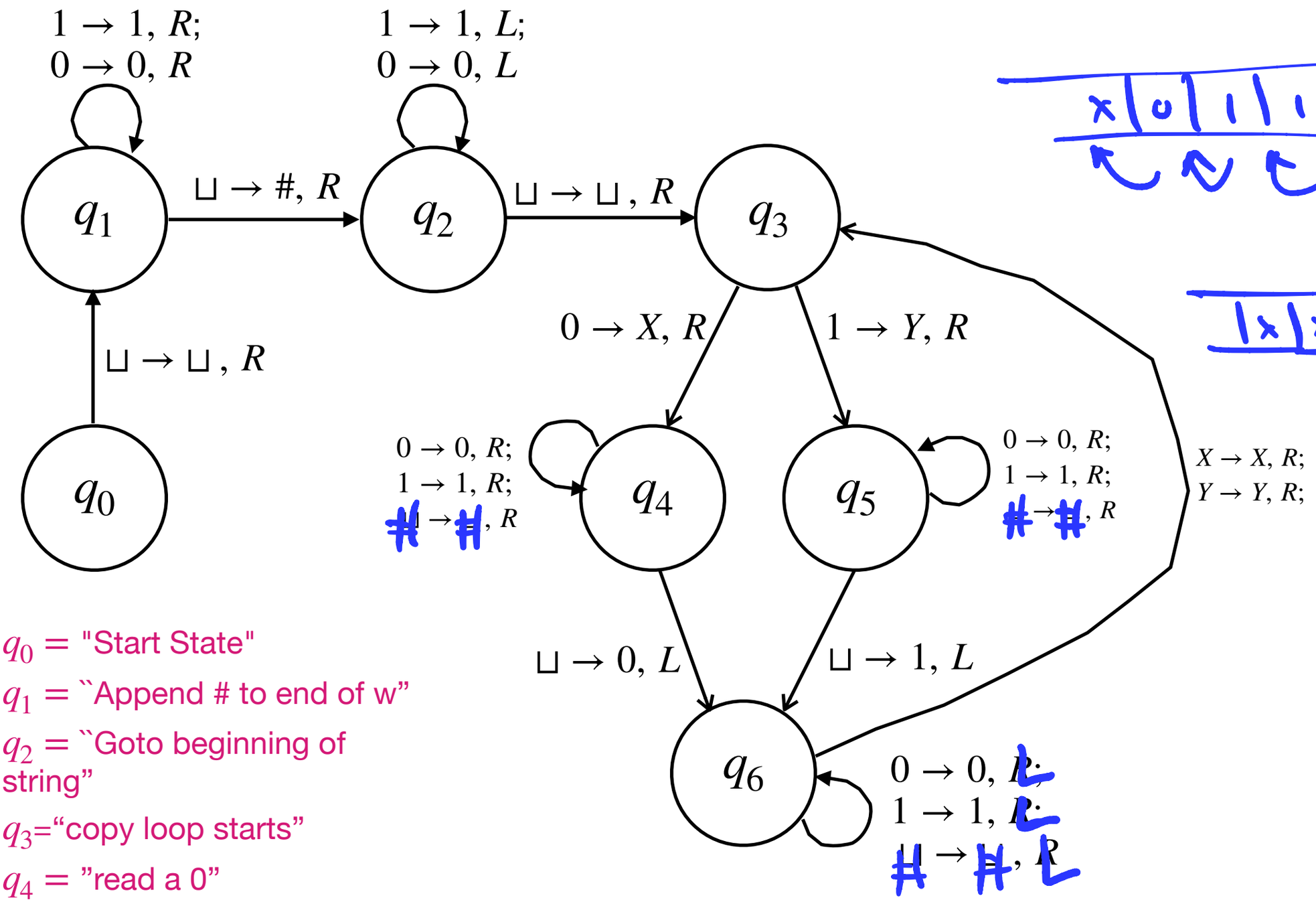

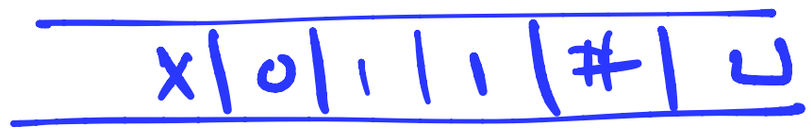

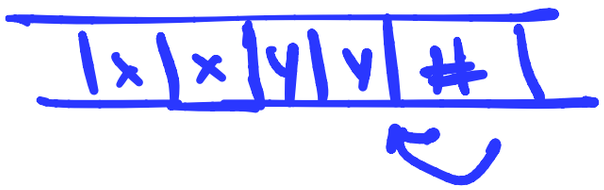

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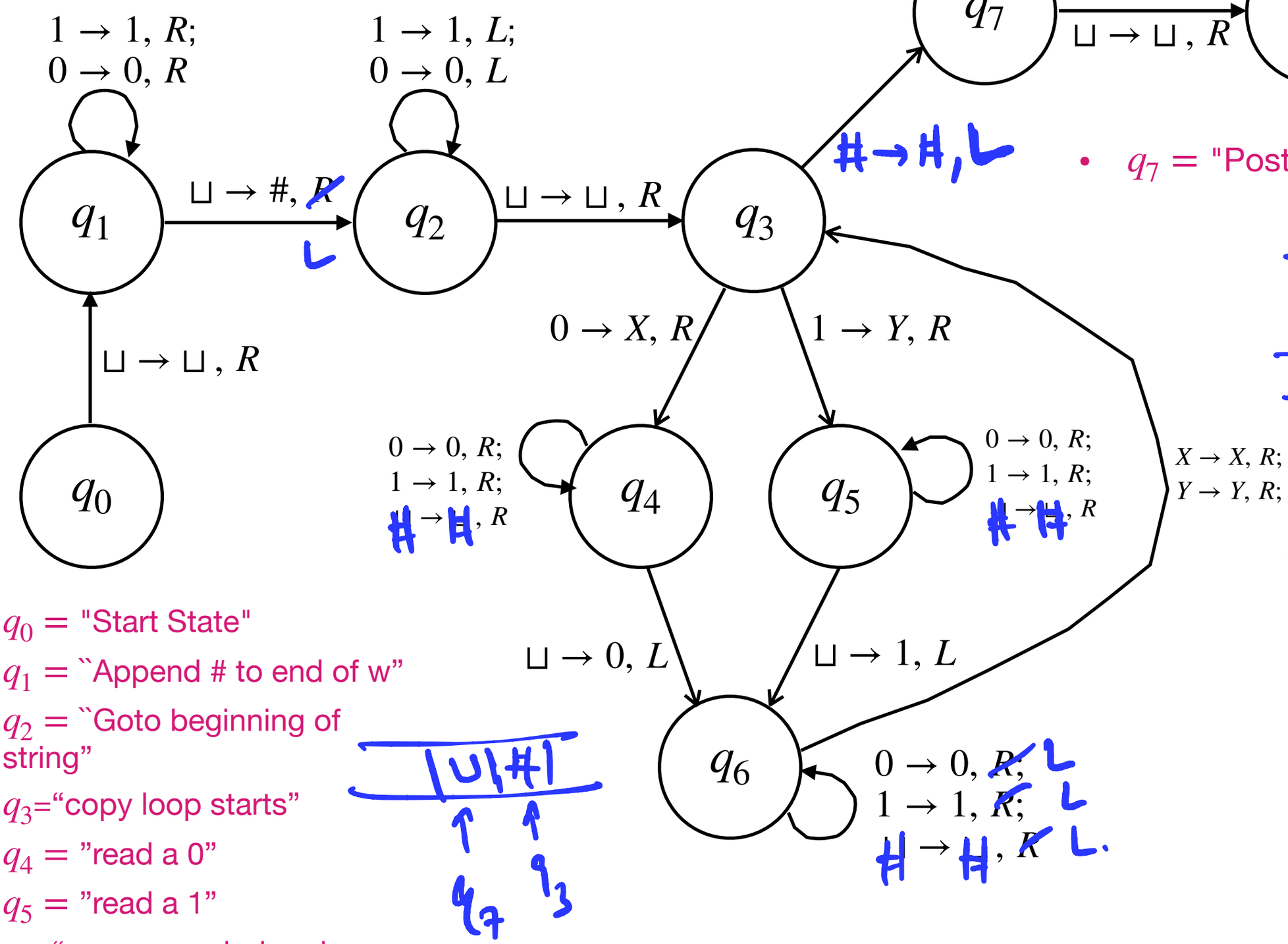

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- q_4 = "read a 0"
- q_5 = "read a 1"
- q_6="copy a symbol and go back to first uncopied symbol"
- q_7 = "Postprocessing"

Complement of a Decidable Language

- The complement of a language L, denoted $\bar{L} = \{x \in \{0,1\}^* \backslash L\}$.
- If L is decidable, then what about $\bar{L}$?

Complement of a Decidable Language

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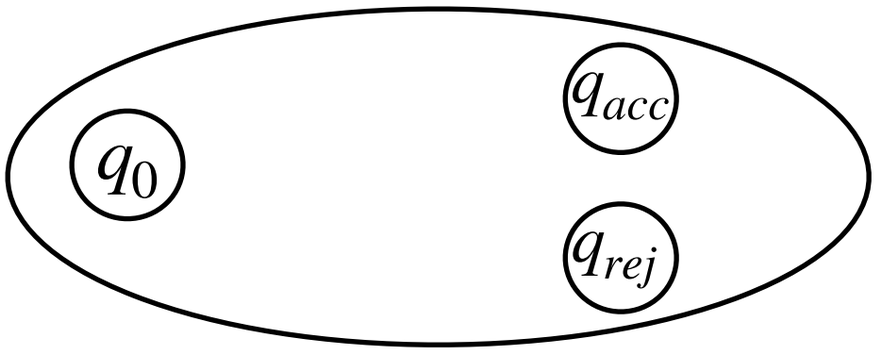
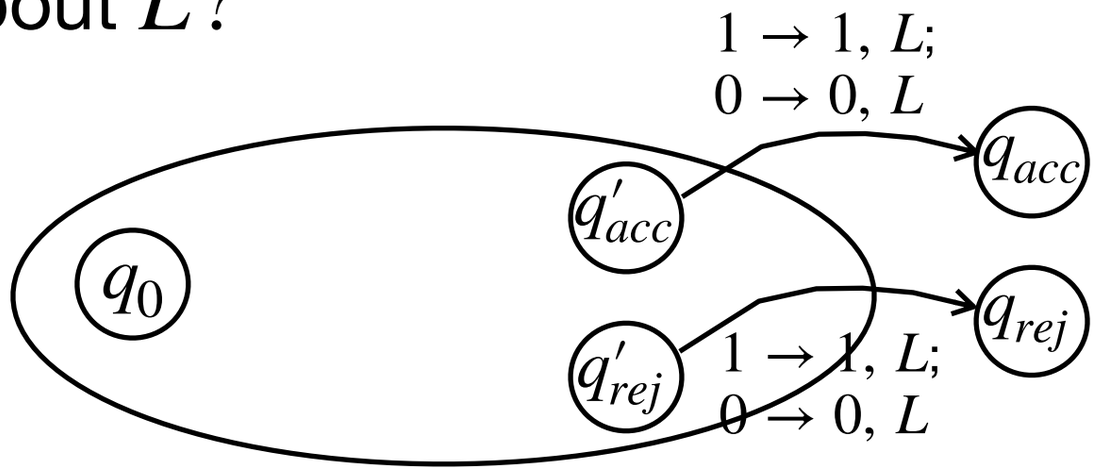


Turing Machine for L



Turing Machine for $\bar{L}$

Complement of a Decidable Language

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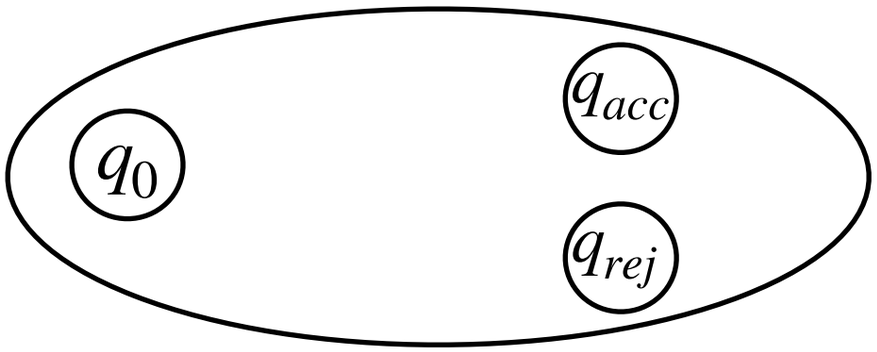
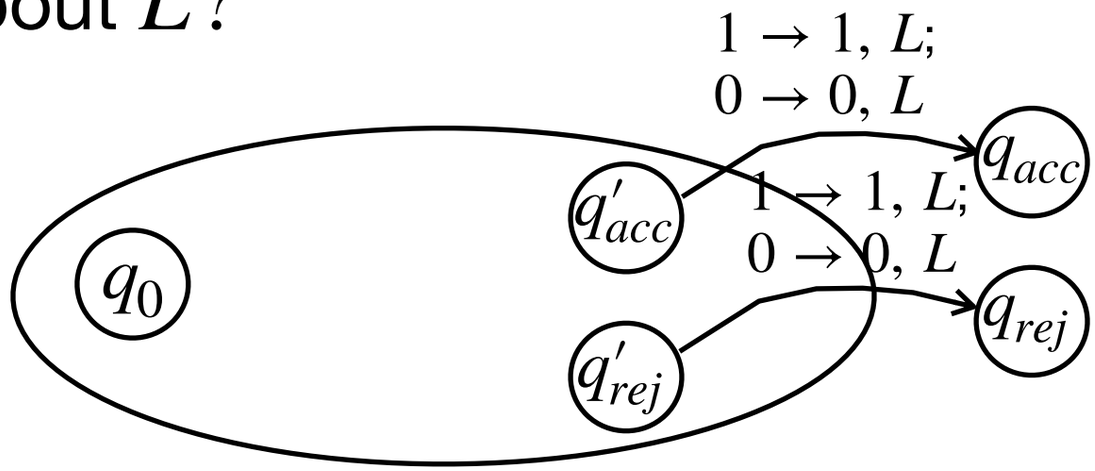




Turing Machine for L

$$M_L = \{Q_L, q_0, q_{acc}, q_{rej}, \Sigma, \delta, \sqcup\}$$

Turing Machine for $\bar{L}$

$$M_{\bar{L}} = \{Q_{\bar{L}}, q_0, q_{acc}, q_{rej}, \Sigma, \delta_{\bar{L}}, \sqcup\}$$

$$Q_{\bar{L}} = Q_L \cup \{q'_{acc}, q'_{rej}\}$$

$$\delta_{\bar{L}} : Q_{\bar{L}} \times \Sigma_{\bar{L}} \to Q_{\bar{L}} \times \Sigma_{\bar{L}} \times \{L, R\}$$

Reverse of a Decidable Language

- Given a decidable language L, is the language $L^R = \{x^R \,|\, x \in L\}$ decidable?

0101 ∈ L then

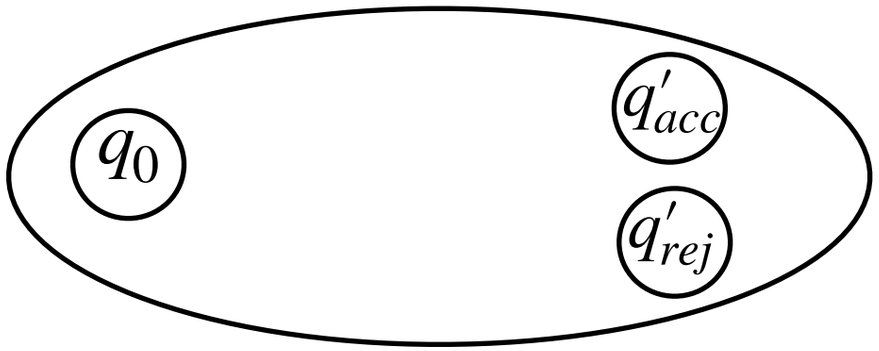
1010 ∈ L^R.

$x = x_1 x_2 \cdots x_n$

$x^R = x_n x_{n-1} \cdots x_1$

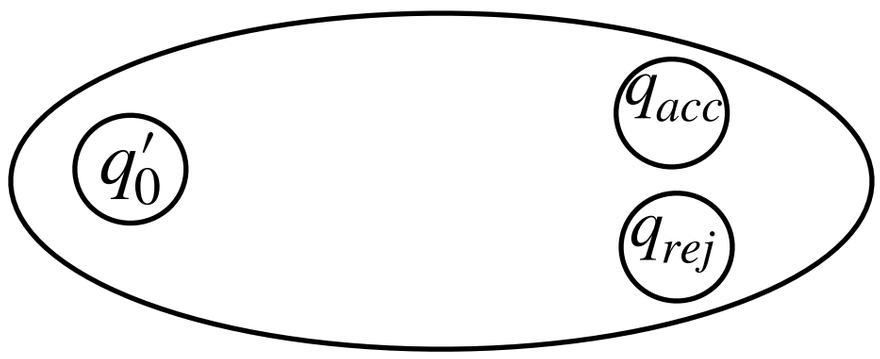
Reverse of a Decidable Language

- Given a decidable language L, is the language $L^R = \{x^R \,|\, x \in L\}$ decidable?



Reverser Turing Machine:

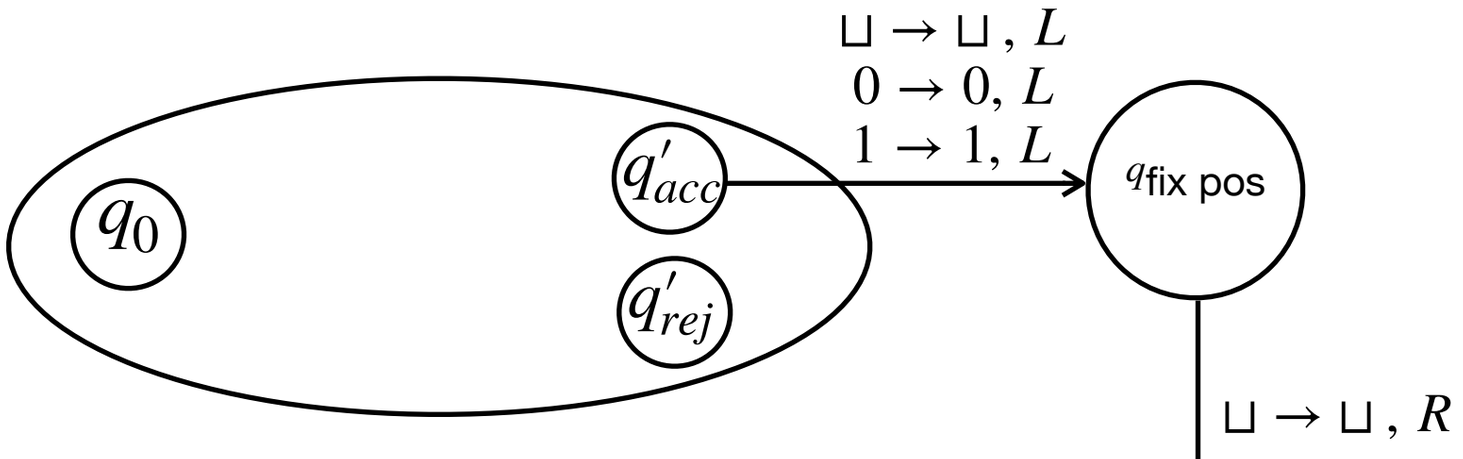
"Reverses a string, places head at the beginning of string and accepts."



Turing Machine deciding L

Reverse of a Decidable Language

- Given a decidable language L, is the language $L^R = \{x^R \mid x \in L\}$ decidable?



Reverser Turing Machine

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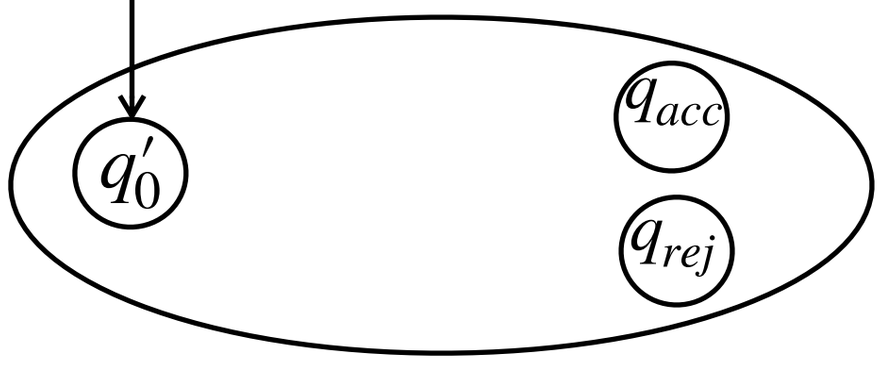

Turing Machine deciding L

A Note on Representation of TMs

- If we formally define a Turing Machine for each task, it can begin to get fairly complicated.

- As we said earlier, Turing machines are a way to mathematically reason about algorithms. Our goal this week and next is to get comfortable with low-level details of Turing machines to convince ourselves that they capture all algorithms.

- Once we are friends with Turing Machines, we will use them in subsequent classes to discuss and reason about computation, but our focus will be on algorithms.

- More examples of this next week.

Properties of Decidable Languages

- **Claim:** Decidable languages are closed under intersection and union.

Summary

- Turing Machine is a 7-tuple $M = (Q, q_{\text{accept}}, q_{\text{reject}}, q_0, \delta, \sqcup, \Sigma)$.
- Computational history of a Turing machine on an input $w \in \{0,1\}^*$ is a series of configurations $C_1, C_2, \cdots, C_T$, where $C_{i+1} = \text{NEXT}(C_i)$.
- Examples of decidable languages and their closure properties.