CSC4010: Introduction to Theoretical Computer Science
Problem Set 1

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Remark: Each problem mentions which lectures it uses techniques from. If you would like feedback on your homework, you can either submit it in class or email it to both of us. If you find any errors, please let us know!

Recap

For arbitrary sets A and B, we use the following notation:

1. We write $|A| = |B|$ if there exists a bijection $f : A \to B$. Equivalently, $|A| = |B|$ if and only if there exists a one-one function $f : A \to B$ and a one-one function $g : B \to A$.
2. We say $|A| \leq |B|$ if there exists a one-one function from A to B.
3. We say $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$. Equivalently, $|A| \leq |B|$ and there is no one-one function from B to A.
4. A set A is *countable* if $|A| \leq |\mathbb{N}|$. It is *countably infinite* if $|A| = |\mathbb{N}|$.

We use $\mathcal{P}(X) = \{A : A \subseteq X\}$ to denote the power set of X. We also write $\{0,1\}^*$ for the set of all finite binary strings, including the empty string ε.

1 Lectures 1 and 2

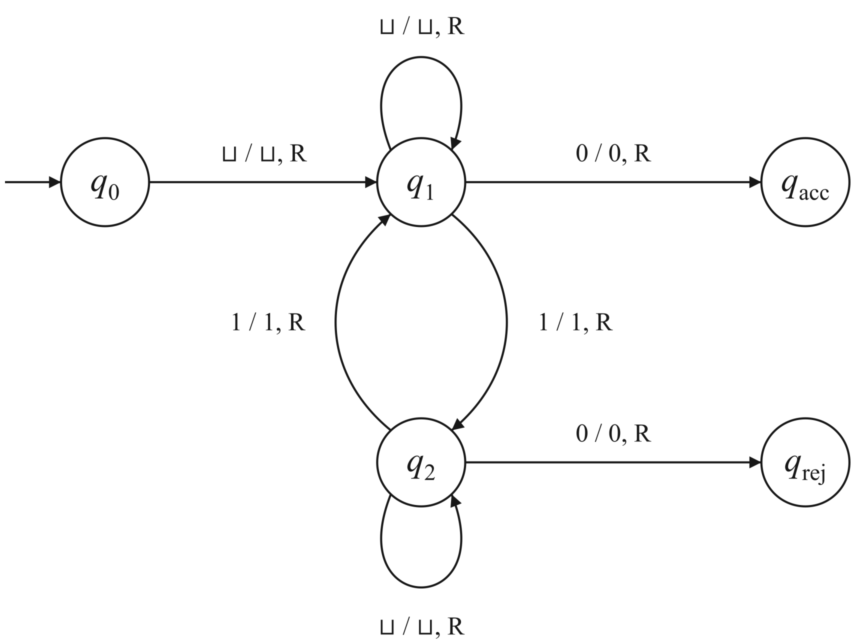
1. In each of the following questions, choose either true or false.
   (a) If $A \subseteq B$, then $|A| \leq |B|$.
   ◯ True
   ◯ False
   (b) If $A \subsetneq B$, then $|A| < |B|$.
   ◯ True
   ◯ False
   (c) Every subset of a countable set is countable.
   ◯ True
   ◯ False
   (d) Every subset of a countably infinite set is countably infinite.
   ◯ True
   ◯ False
   (e) Every language $L \subseteq \{0,1\}^*$ is countable.
   ◯ True
   ◯ False
   (f) The set of all languages $\mathcal{L} = \{L : L \subseteq \{0,1\}^*\}$ is countable.
   ◯ True
   ◯ False

2. Let $\mathcal{S} = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$ be the set of all finite sequences of non-negative integers, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The case $k = 0$ corresponds to the empty sequence. Show that $\mathcal{S}$ is countable using one of the following methods. You may use the fact that $\{0, 1\}^*$ is countable.

   1. Construct a one-one function
   $$f : \mathcal{S} \longrightarrow \{0, 1\}^*.$$
   Make sure to explain why your function is one-one.
   2. Give an enumeration of $\mathcal{S}$ in which every element has a finite position.
   Hint. For example, you may organize the sequences according to the value
   $$k + n_1 + \cdots + n_k$$
   for a sequence $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

3. For an arbitrary set X, let $\mathcal{P}(X) = \{A : A \subseteq X\}$ denote the power set of X. Prove the following.
   (a) Show that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.
   (b) Prove that there is no bijection $f : \mathcal{P}(X) \longrightarrow X$. Conclude that $|\mathcal{P}(X)| > |X|$.
   Hint: Prove by contradiction. Assume f is a bijection. Define $D_f \in \mathcal{P}(X)$ by $D_f = \{x \in X \mid x \notin f^{-1}(x)\}$.

2 Lectures 3 and 4



1. Consider the Turing machine M shown above. Its input alphabet is $\{0, 1\}$ and its tape alphabet is $\{0, 1, \sqcup\}$, where $\sqcup$ denotes the blank symbol.

   The initial state is q_0. As in class, the head starts on the blank cell immediately to the left of the input. The machine halts immediately upon entering q_{acc} or q_{rej}. An edge label $a/b, D$ means: read a, write b, move in direction $D \in \{L, R\}$, and enter the destination state. Answer the following.

(a) Use natural language or pseudocode to describe the behavior of the machine.

(b) Write down the computational history of the machine on each of the following inputs:

- '101'
- '110'
- '111'

(c) Does this machine recognize a language? Give a brief explanation.

(d) Does this machine decide a language? Give a brief explanation.

2. Design Turing machines for the following languages. For each of the following languages, describe a Turing machine that can decide it using natural language or pseudocode. Then **choose one** of the three languages and formally design your machine. Explain how your machine is implemented (for example, what the states are doing, whether you use extra tape symbols, and why the machine accepts exactly the intended strings).

   (a) All nonempty binary strings whose first and last symbols are the same.

   (b) All binary strings containing an equal number of 0s and 1s.

   (c) All nonempty binary strings that represent a multiple of 3 when interpreted as binary numbers. Leading zeros are allowed.