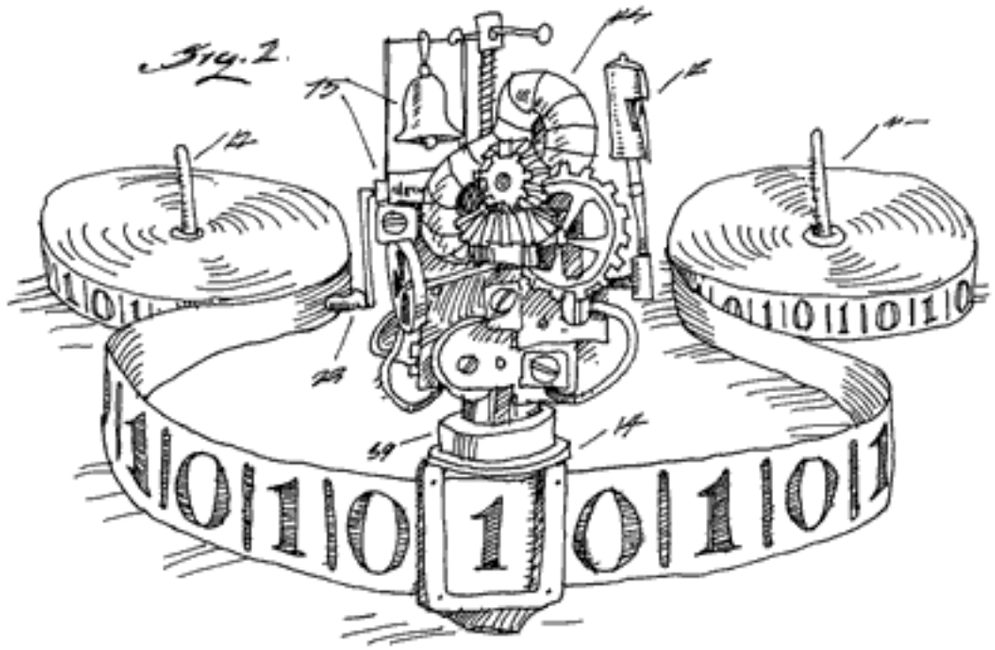
CSC4010: Introduction to Theoretical Computer Science



Lecture 9

Adapted from Sipser's MIT OCW Course on TOC

香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

SCHOOL OF DATA SCIENCE
數據科學學院

SDS

TOPICAL SEMINAR SERIES



Speaker

He SUN

Director, Center for Algorithms and Learning Theory
Shenzhen Institute of Advanced Technology, CAS

Associate Director, State Key Laboratory of Biomedical Imaging Science and System

Faster Dynamic Graph Clustering with Hierarchical Graph Contraction

Abstract

We study clustering algorithm for dynamic evolving graphs G_t, in which new edges (and potentially new vertices) are added into a graph, and the underlying cluster structure of the graph gradually changes over time. By designing a hierarchical data structure for dynamically maintaining the cluster structure of G_t we prove that the cluster structure of G_T can be maintained with O(d) amortised update time and O(n_T^{1/d}) amortised query time, for an arbitrary large constant d. We further demonstrate this improvement with experiments on both synthetic and real-world datasets.

The talk is based on joint work with Suranjan De (University of Edinburgh) and Steinar Leanen (Google Zurich). The work appears at NeurIPS 2026.

About the Speaker

Professor He Sun is the Director of Center of Algorithms and Learning Theory, Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, and the Associate Director of the State Key Laboratory of Biomedical Imaging Science and System with over 150 researchers. He received his Ph.D. from Fudan University in 2010 and worked at the Max Planck Institute for Informatics, UC Berkeley, University of Bristol, and University of Edinburgh. He is a recipient of the Chinese High-Level Talent Recruitment Program for Overseas Experts and is a fellow selected under the CAS 100 Talents Program (Category A).

He has served as an area chair and PC member of leading international conferences like ICML (2026), NeurIPS (2026), ICLR (2027), and STOC (2026), and as regular reviewer for research councils including ERC, UKRI, and ISF.

Host / Aditi Dudeja

Assistant Professor
School of Data Science
CUHK-Shenzhen

10.15 2026

BEIJING TIME	**11:00** AM– **12:00** NOON
VENUE	**Room 413** **Zhi Xin Building**
LANGUAGE	**English**

SDS Wechat



Undecidability and Unrecognizability

1) $A_{TM} = \{ \langle M, w \rangle \mid M \text{ accepts } w \}$

$L_{self} = \{ \langle M \rangle \mid M \text{ rejects } \langle M \rangle \}$

$HALT_{TM} = \{ \langle M, w \rangle \mid M \text{ halts on } w \}$

(2) Diagonalization : A_{TM}, L_{self}.

(3) Reductions: To show undecidability for $HALT_{TM}$.

A_{TM} can be reduced $HALT_{TM}$

$\Rightarrow$ If an algorithm M_H which decides $HALT_{TM}$, then, we can use that as a subroutine to decide A_{TM}.

$\Rightarrow$ As A_{TM} is undecidable $\Rightarrow$ $HALT_{TM}$ is also undecidable

Mapping Reducibility

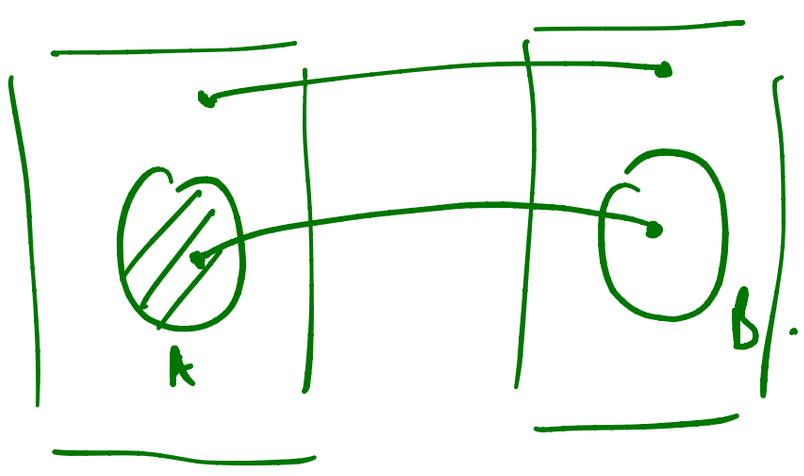
Mapping Reducibility

- **Definition:** A function $f : \Sigma^* \to \Sigma^*$ is said to be **computable** if there exists a Turing Machine F, which on input $w \in \Sigma^*$ terminates with just $f(w)$ written on its tape, for any w.

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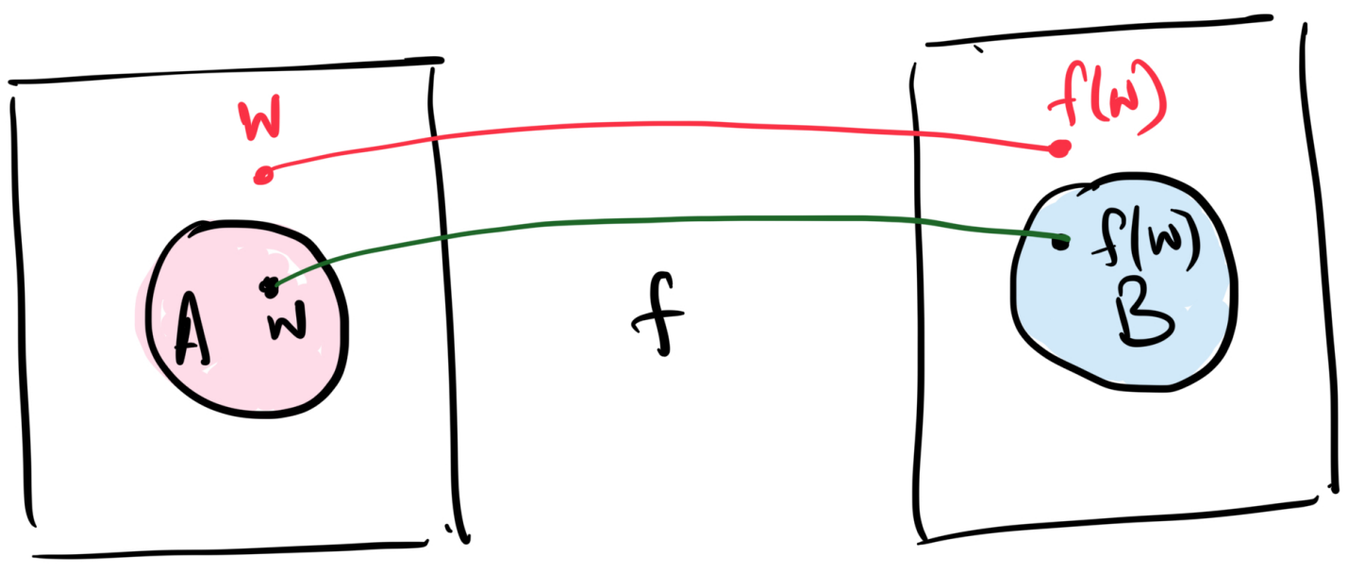
- **Definition:** A language A is **mapping reducible** to a language B (denoted $A \leq_m B$) if and only if there exists a computable function $f : \Sigma^* \to \Sigma^*$ such that for every $w \in \Sigma^*$, $w \in A \iff f(w) \in B$.



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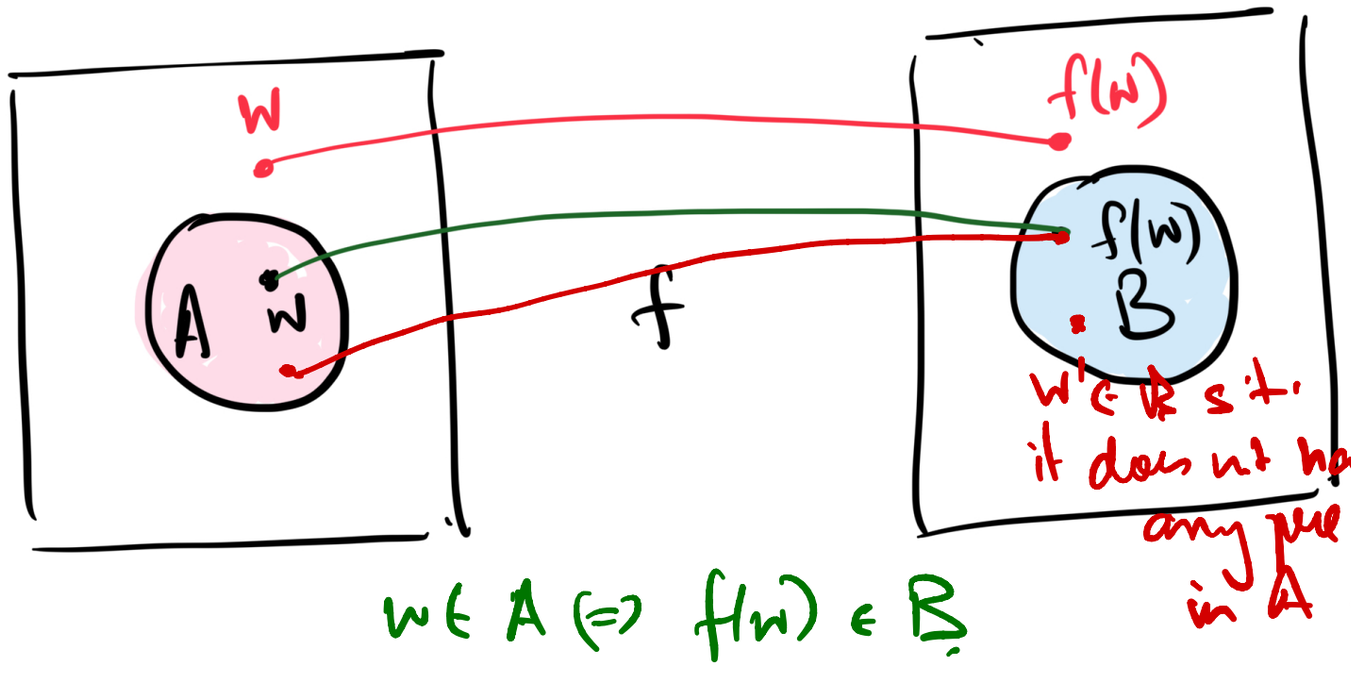
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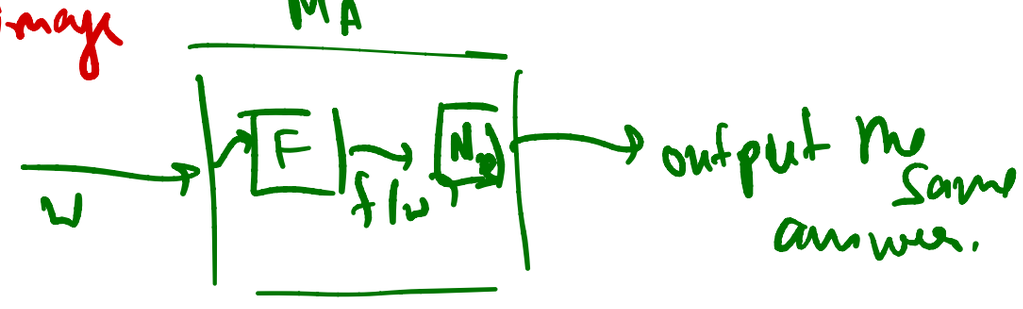
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f is called a mapping reduction from A to B .



$A_{TM} \leq_m \overline{\text{EMPTY}}_{TM}$ $\longmapsto$ $\{\langle M \rangle \mid L(M) \neq \emptyset\}$

$A_{TM} \leq_m \overline{\text{EMPTY}}_{TM}$

↪ {⟨M,w⟩ | M accepts w}.

↦ {⟨M⟩ | L(M) ≠ ∅}

Solution: Given input $\langle M, w \rangle$ for A_{TM}, $f(\langle M, w \rangle) = \langle M_w \rangle$ defined as follows:

* f should be computable
* ⟨M,w⟩ ∈ A_{TM} ⇔ ⟨M_w⟩ ∈ $\overline{\text{EMPTY}}_{TM}$

$A_{TM} \leq_m \overline{\text{EMPTY}}_{TM}$

Solution: Given input $\langle M, w \rangle$ for A_{TM}, $f(\langle M, w \rangle) = \langle M_w \rangle$ defined as follows:

$M_w =$ "On input $x \in \{0,1\}^*$

1. If $x \neq w$, then reject.
2. Simulate M on w.

$\langle M, w \rangle$ it is possible to "algorithmically" compute $\langle M_w \rangle$

Auxiliary states: (1) to check if $x = w$ or not?
(2) then simulate M on w.

⇓

taking description of M, and adding states.
(2) Adding new transitions.

Therefore, given description of M, w we can come up with $\langle M_w \rangle$.

(1) f is membership/non-membership preserving??

$A_{TM} \leq_m \overline{\text{EMPTY}}_{TM}$

Solution: Given input $\langle M, w \rangle$ for A_{TM}, $f(\langle M, w \rangle) = \langle M_w \rangle$ defined as follows:

$M_w =$ "On input $x \in \{0,1\}^*$

1. If $x \neq w$, then reject.
2. Simulate M on w.

M_w also accepts w.

$$L(M_w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ does not accept } w \end{cases}$$

We can show that f is membership/non-membership preserving:

(1) $L(M_w) \neq \emptyset \iff M$ accepts w

$\iff \langle M, w \rangle \in A_{TM}$

$\Updownarrow$

$\langle M_w \rangle \in \overline{\text{EMPTY}}_{TM}$

f: mapping reduction from A_{TM} to $\overline{\text{EMPTY}}_{TM}$.

$\Rightarrow$ A_{TM} is undecidable $\Rightarrow$ $\overline{\text{EMPTY}}_{TM}$ is also undecidable.

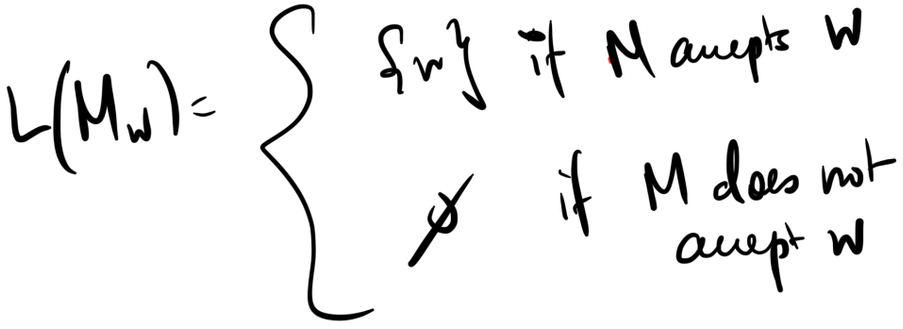
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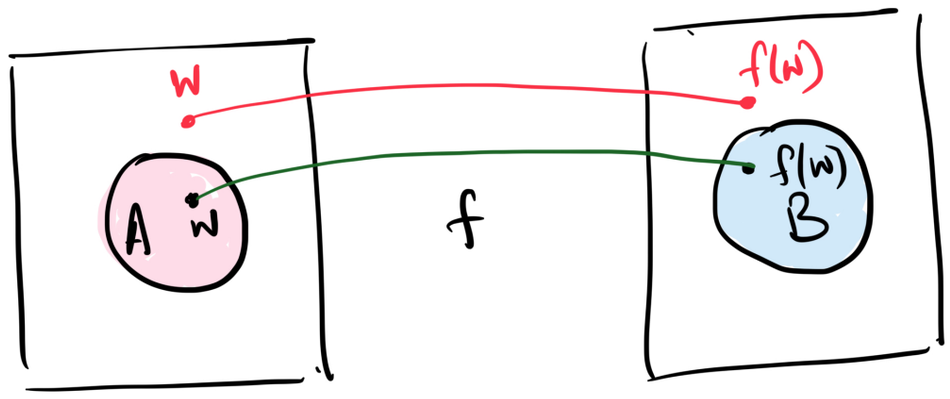
1. If $x \neq w$, then reject.
2. Simulate M on w .

Claim: $f(\langle M, w \rangle) \in \overline{\text{EMPTY}}_{TM} \iff \langle M, w \rangle \in A_{TM}$.

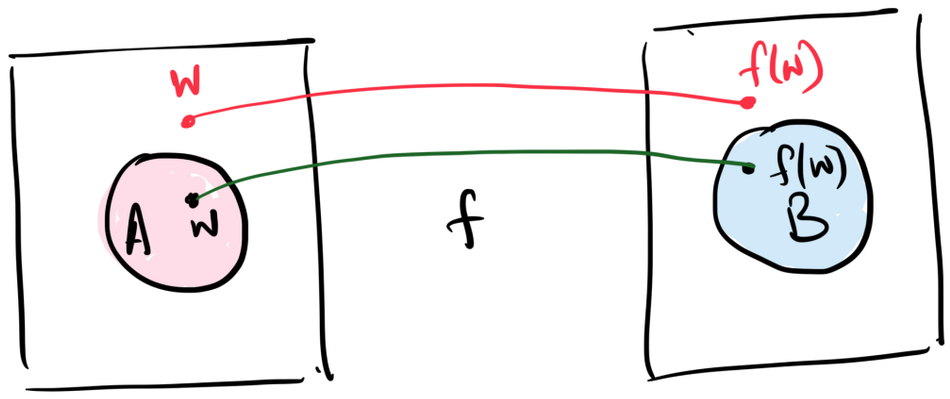


Mapping Reducibility: Properties

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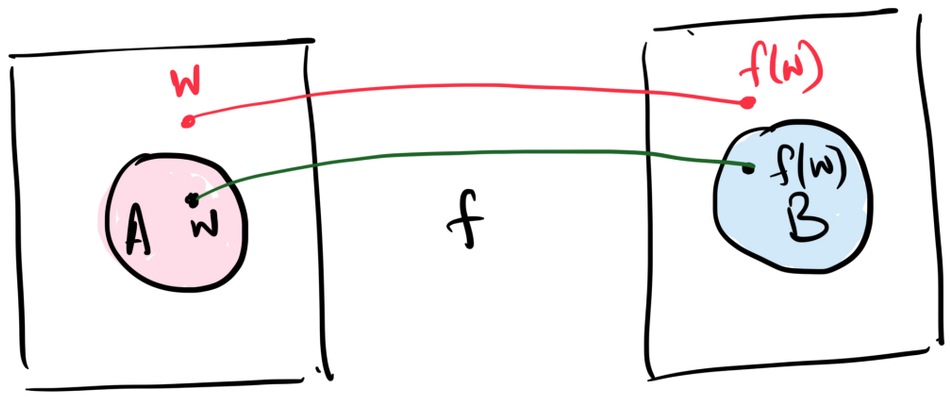


Mapping Reducibility: Properties



Theorem: If $A \leq_m B$ and B is decidable (recognizable), then A is decidable (respectively, recognizable).

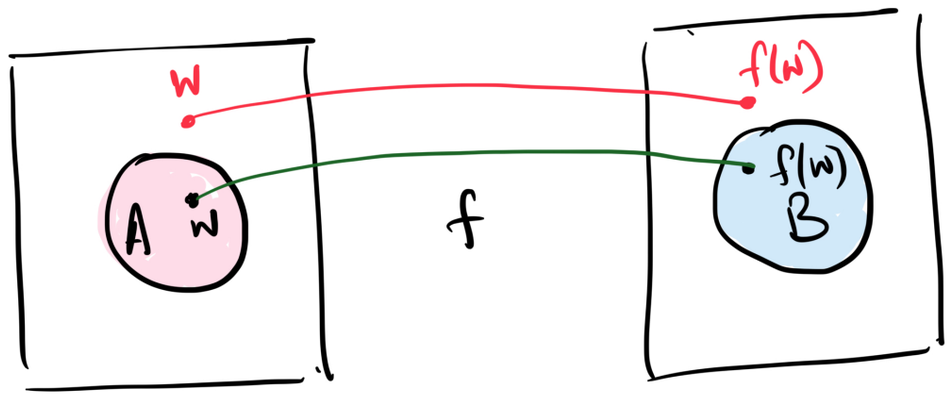
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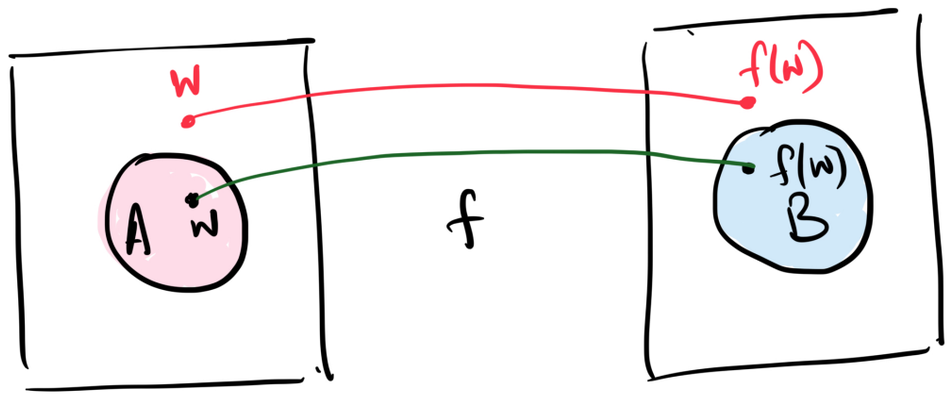


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- **A recipe for undecidability/unrecognizability:** To show B is undecidable (unrecognizable), find undecidable (respectively, unrecognizable) A and a mapping reduction from A to B .

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- **A recipe for undecidability/unrecognizability:** To show B is undecidable (unrecognizable), find undecidable (respectively, unrecognizable) A and a <u>mapping reduction</u> from A to B.

→ The same f also preserves non-membership/membership in $\bar{A}$ & $\bar{B}$.

Theorem: If $A \leq_m B$ then $\bar{A} \leq_m \bar{B}$.

$w \in A \Leftrightarrow f(w) \in B$
$\Rightarrow w \in \bar{A} \Leftrightarrow f(w) \in \bar{B}$.

Using Mapping Reduction for Unrecognizability

EMPTY_{TM} is unrecognizable

- Let $\text{EMPTY}_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$.

Theorem: EMPTY_{TM} is unrecognizable.

$\overline{A_{TM}} \leq \text{EMPTY}_{TM}$

i.e. $\{\langle M, w \rangle \mid M \text{ does not accept } w\}$

EMPTY$_{TM}$ is unrecognizable

- Let EMPTY$_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$.

Theorem: EMPTY$_{TM}$ is unrecognizable.

- **Proof:** Proof by contradiction. We will show $\bar{A}_{TM} \leq_m$ EMPTY$_{TM}$.

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$L(M_w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \varnothing & \text{if } M \text{ does not accept } w \end{cases}$

$\langle M, w \rangle \in \bar{A}_{TM} \iff \langle M_w \rangle \in \overline{\text{EMPTY}}_{TM}$

$L(M_w) = \varnothing$

f : a mapping reduction.

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$$\begin{aligned} f(\langle M, w \rangle) \in \text{EMPTY}_{TM} &\iff \langle M_w \rangle \in \text{EMPTY}_{TM} \\ &\iff L(M_w) = \varnothing \\ &\iff M \text{ does not accept } w \\ &\iff \langle M, w \rangle \in \bar{A}_{TM} \end{aligned}$$

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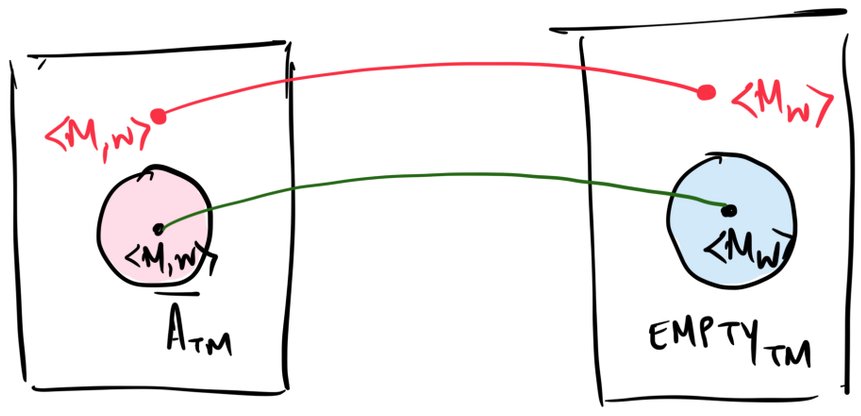
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$f(\langle M, w \rangle) \in$ EMPTY$_{TM}$ $\iff$ $\langle M_w \rangle \in$ EMPTY$_{TM}$
$\iff$ $L(M_w) = \varnothing$
$\iff$ M does not accept w
$\iff$ $\langle M, w \rangle \in \bar{A}_{TM}$



More Examples

EQ_{TM} and $\overline{EQ}_{TM}$ are unrecognizable

- Let $EQ_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$.

Theorem: EQ_{TM} is unrecognizable.

$\overline{A}_{TM} \leq_m EQ_{TM}$

$\Downarrow$

$\{\langle M, w\rangle \mid M \text{ does not accept } w\}$

$f(\langle M, w\rangle) = \langle M_1, M_2\rangle$

$\langle M, w\rangle \in \overline{A}_{TM} \Leftrightarrow L(M_1) = L(M_2)$

$(M_1, M_2) \in EQ_{TM}$

EQ_{TM} and $\overline{\mathrm{EQ}}_{TM}$ are unrecognizable

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Theorem: EQ_{TM} is unrecognizable.

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Theorem: EQ_{TM} is unrecognizable.

- **Proof:** Proof by contradiction. We will show $\bar{A}_{TM} \leq_m EQ_{TM}$.

Reduction Function: $f(\langle M, w\rangle) = \langle T_w, M_\emptyset\rangle$ $\overline{A}_{TM} \leq_m EQ_{TM}$

$= \{\langle M_1, M_2\rangle \mid M \text{ does not accept } w\}$

$f(\langle M, w\rangle) = \langle M_1, M_2\rangle$

$\langle M, w\rangle \in \overline{A}_{TM} \Leftrightarrow L(M_1) = L(M_2)$

$(M_1, M_2) \in EQ_{TM}$

T_w on i/p x:
(1) Ignores x.
(2) Runs M on w.

reject all input.

$M_\emptyset$: Rejects all input.

$L(T_w) = \begin{cases} \{0,1\}^* & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ not accept } w \end{cases}$

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Reduction Function: $f(\langle M, w\rangle) = \langle T_w, M_\emptyset\rangle$

T_w = "On input $x \in \{0,1\}^*$

1. Ignore x.
2. Simulate M on w.

Claim: f preserves $f(\langle M,w\rangle) \in EQ_{TM} \iff \langle M,w\rangle \in \overline{A}_{TM}$

EQ_{TM} and $\overline{\text{EQ}}_{TM}$ are unrecognizable

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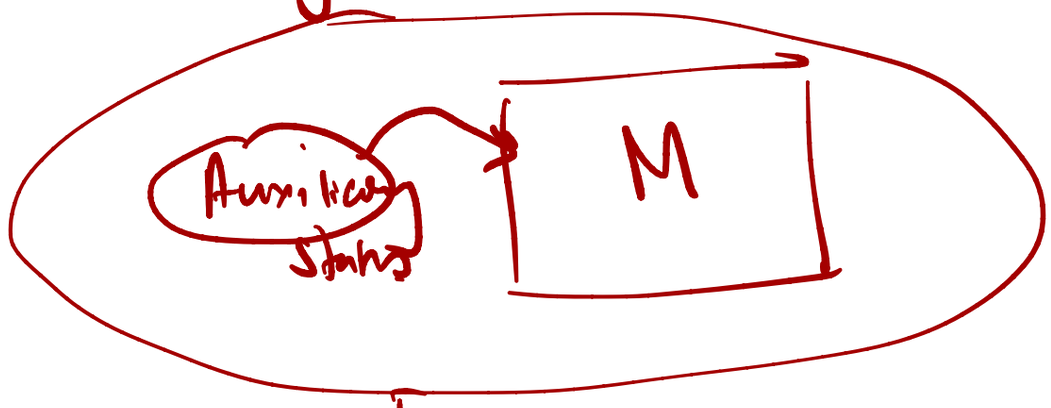
- **Proof:** Proof by contradiction. We will show $\bar{A}_{TM} \leq_m \text{EQ}_{TM}$.

Reduction Function: $f(\langle M, w\rangle) = \langle T_w, M_\emptyset\rangle$

$T_w =$ "On input $x \in \{0,1\}^*$

1. Ignore x .
2. Simulate M on w .

T_w may not halt as M may not halt.



F

F always halts!

1) F just adds extra states + description how to go from Auxiliary States to M

2) This doesn't depend on M or w.

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Theorem: EQ_{TM} is unrecognizable.

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Reduction Function: $f(\langle M, w\rangle) = \langle T_w, M_\varnothing\rangle$

$T_w =$ "On input $x \in \{0,1\}^*$

1. Ignore x .
2. Simulate M on w .

$L(T_w) = \begin{cases} \{0,1\}^* & \text{if } M \text{ accepts } w \\ \varnothing & \text{if } M \text{ rejects } w \end{cases}$

$$\begin{aligned} \langle T_w, M_\varnothing\rangle \in \mathrm{EQ}_{TM} &\iff L(T_w) = L(M_\varnothing) \\ &\iff L(T_w) = \varnothing \\ &\iff M \text{ does not accept } w \\ &\iff \langle M, w\rangle \in \bar{A}_{TM} \end{aligned}$$

EQ_{TM} and $\overline{\mathrm{EQ}}_{TM}$ are unrecognizable

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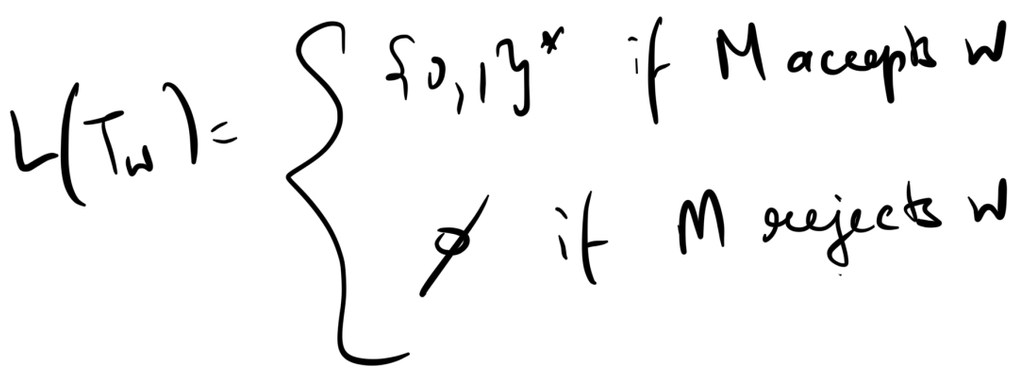
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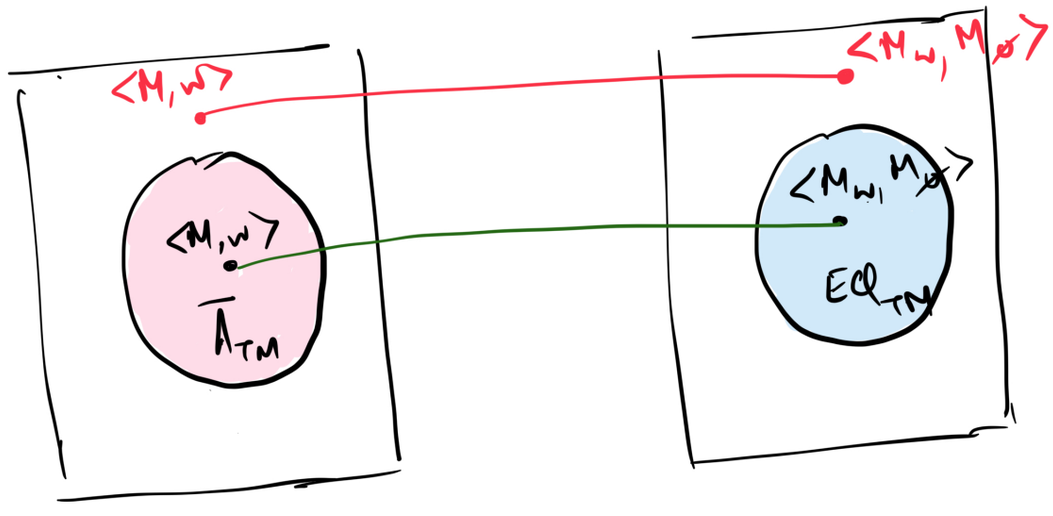
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$\langle T_w, M_\emptyset\rangle \in \mathrm{EQ}_{TM} \iff L(T_w) = L(M_\emptyset)$
$\iff L(T_w) = \emptyset$
$\iff M$ does not accept w
$\iff \langle M, w\rangle \in \bar{A}_{TM}$



EQ_{TM} and $\overline{EQ}_{TM}$ are unrecognizable

- Let $\overline{EQ}_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) \neq L(M_2)\}$.

Theorem: $\overline{EQ}_{TM}$ is unrecognizable.

$\overline{A}_{TM} \leq_m \overline{EQ}_{TM}$

$M_1 =$ T_w: on input x
(a) $x \neq w$ then, it rejects.
(b) Simulates M on w.

$\langle M, w\rangle$ $\qquad$ $\langle M_1, M_2\rangle$

$\langle M, w\rangle \in \overline{A}_{TM} \Leftrightarrow \langle M_1, M_2\rangle \in \overline{EQ}_{TM}$.

$M_2 = ?$ $\qquad$ M: on input x:
(a) $x \neq w$ then, it rejects
(b) else it accepts.

$L(T_w) = \{w\}$ when $\langle M, w\rangle \notin \overline{A}_{TM}$

$\emptyset$ when $\langle M, w\rangle \in \overline{A}_{TM}$

$f(\langle M, w\rangle) = \langle T_w, M_\emptyset\rangle$

Claim: f is computable & f preserves membership.

$\langle M, w\rangle \in \overline{A}_{TM} \Leftrightarrow$ M does not accept w
$\Leftrightarrow L(T_w) = \emptyset$
$\Leftrightarrow L(T_w) \neq \{w\}$

$\Leftrightarrow \langle T_w, M\rangle \in \overline{EQ}_{TM}$
$L(T_w) \neq L(M)$

EQ_{TM} and $\overline{\mathrm{EQ}}_{TM}$ are unrecognizable

- Let $\overline{\mathrm{EQ}}_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) \neq L(M_2)\}$.

Theorem: $\overline{\mathrm{EQ}}_{TM}$ is unrecognizable.

- **Proof:** Proof by contradiction. We will show $\overline{A}_{TM} \leq_m \overline{\mathrm{EQ}}_{TM}$.

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$$f(\langle T_w, M_{acc}\rangle) \in \overline{EQ}_{TM} \iff L(T_w) \neq L(M_{acc})$$
$$\iff L(T_w) \neq \{0,1\}^*$$
$$\iff M \text{ does not accept } w$$
$$\iff \langle M, w\rangle \in \bar{A}_{TM}$$

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- Let $\overline{\mathrm{EQ}}_{TM} = \{\langle M_1, M_2\rangle \mid L(M_1) \neq L(M_2)\}$.

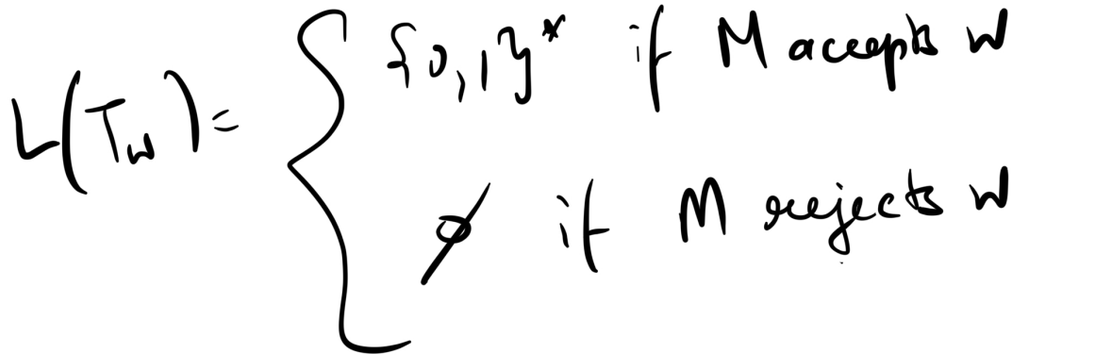
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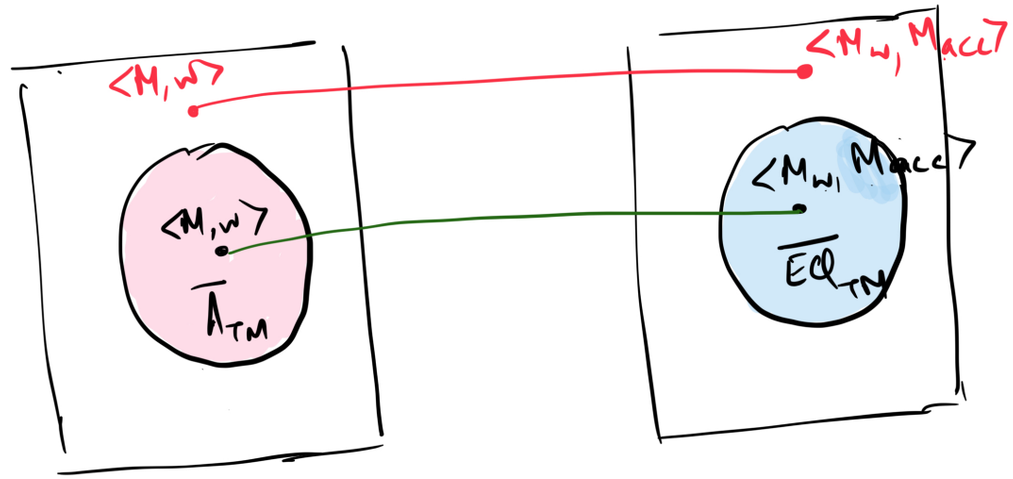


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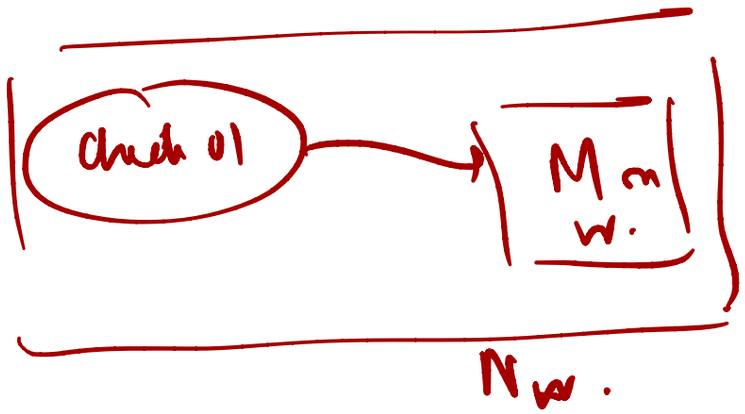
$\iff \langle M, w\rangle \in \bar{A}_{TM}$



L_{01} is unrecognizable

- Let $L_{01} = \{\langle M \rangle \mid L(M) = \{01\}\}$.

Theorem: L_{01} is unrecognizable.



$\overline{A}_{TM} \leq_m L_{01}$

$\langle M, w \rangle \longrightarrow \langle N_w \rangle$.

N_w on input x:
(1) $x = 01$ then accept.
(2) $x \neq 01$ then, simulate M on w.

$L(N_w) =$ $\{01\}$ if M does not accept w.
Σ^* if M accepts w.

$\langle M \rangle \in L_{01} \Leftrightarrow \langle M, w \rangle \in \overline{A}_{TM}$

f: reverses membership.

$\langle M \rangle \in L_{01} \Rightarrow \langle M, w \rangle \in \overline{A}_{TM}$

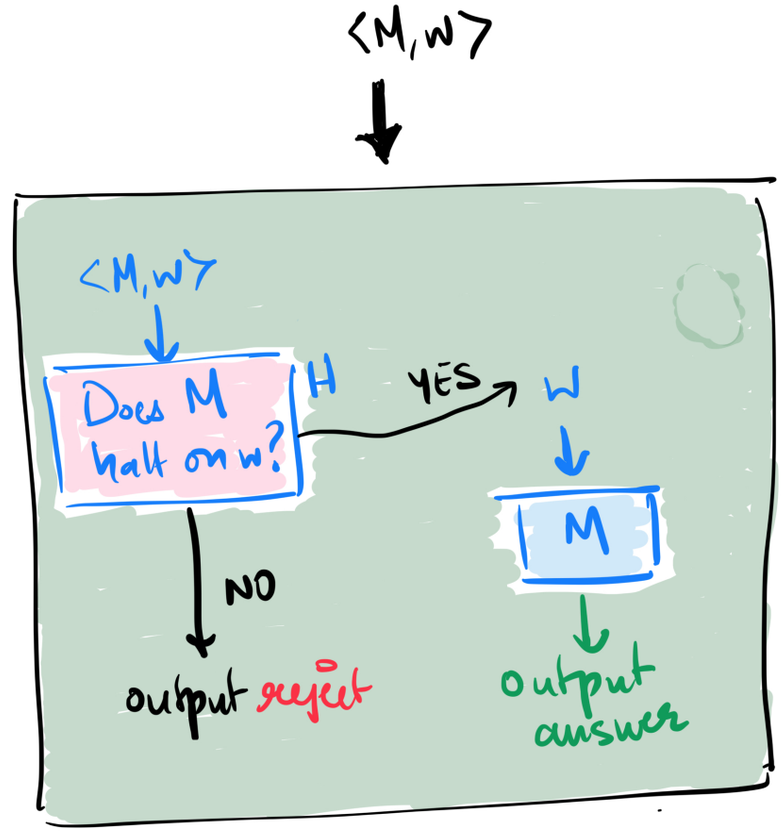
$\langle M \rangle \notin L_{01} \Rightarrow \langle M, w \rangle \notin \overline{A}_{TM}$.

Turing Reducibility vs Mapping Reducibility

- We already prove that A_{TM} is undecidable (by diagonalization).
- Our goal was to show a candidate language L is undecidable.
- We showed if L is decidable, then so is A_{TM} .
  - Design an algorithm for A_{TM} using the hypothetical algorithm for L .
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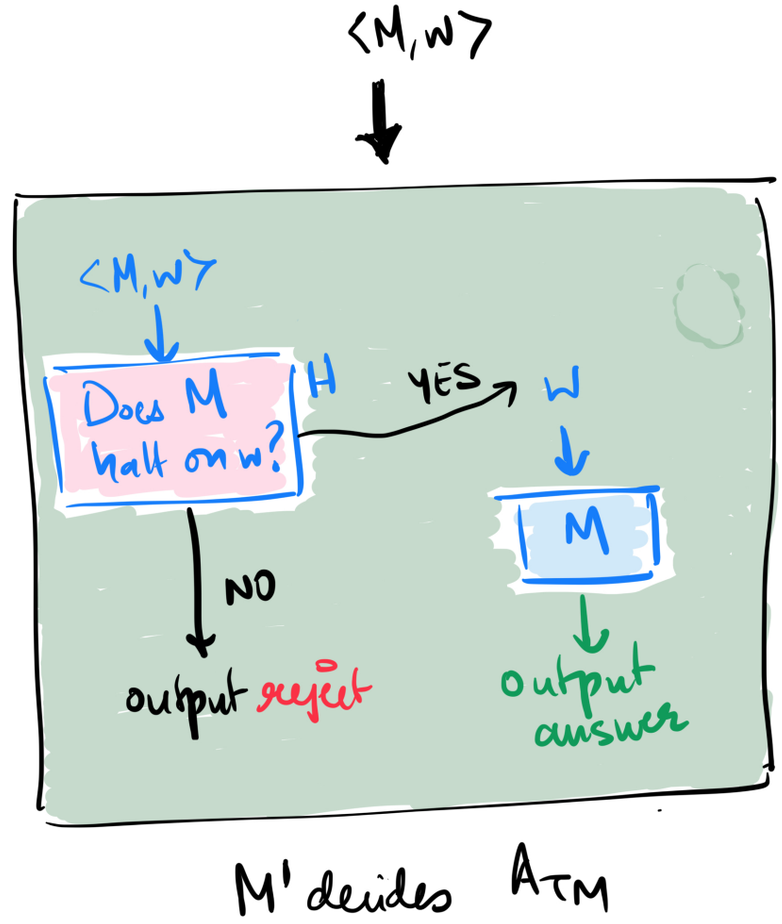


M' decides A_{TM}

The algorithm for $HALT_{TM}$ is being used as a subroutine for A_{TM}.

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Proving HALT$_{TM}$ is undecidable.

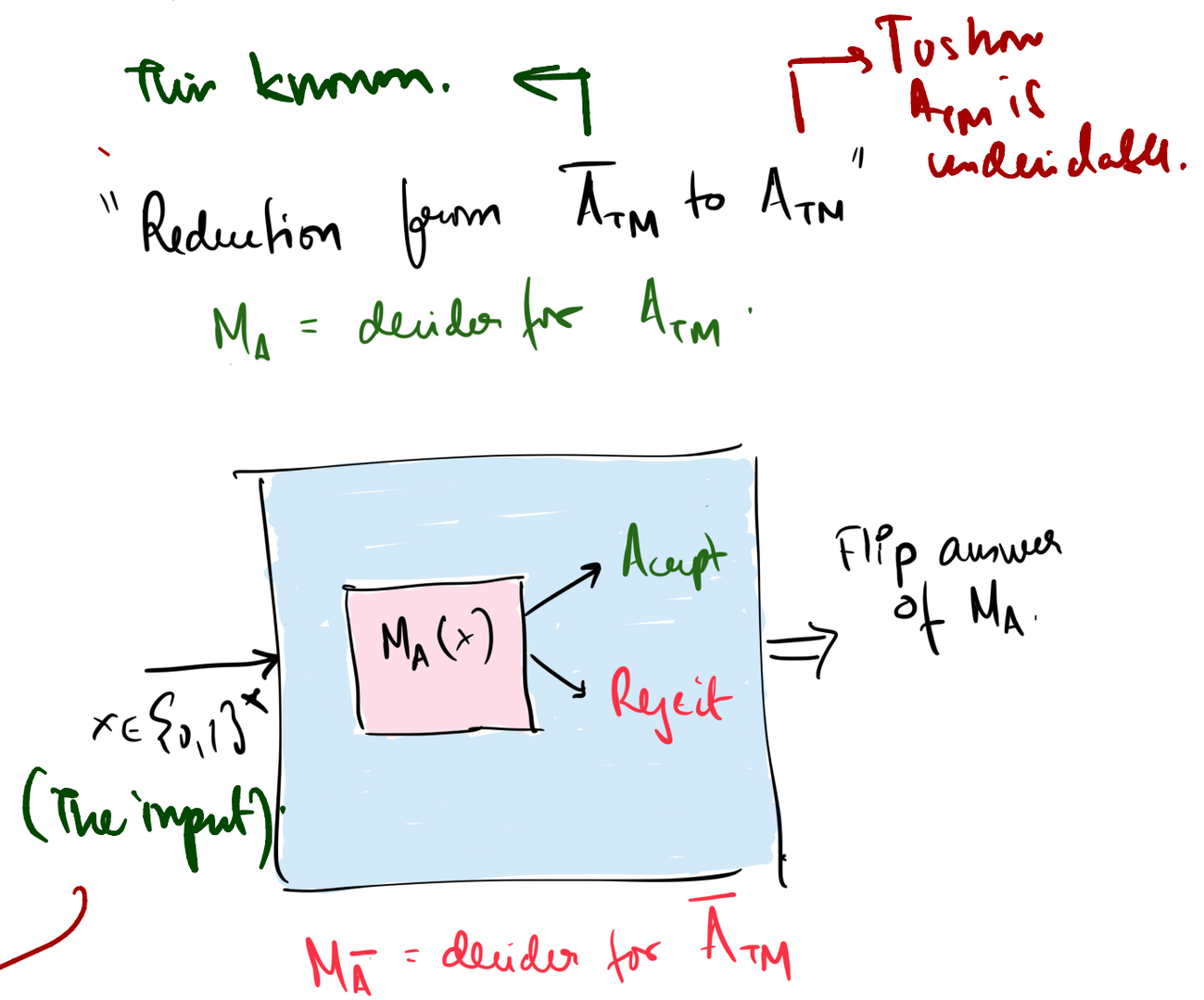
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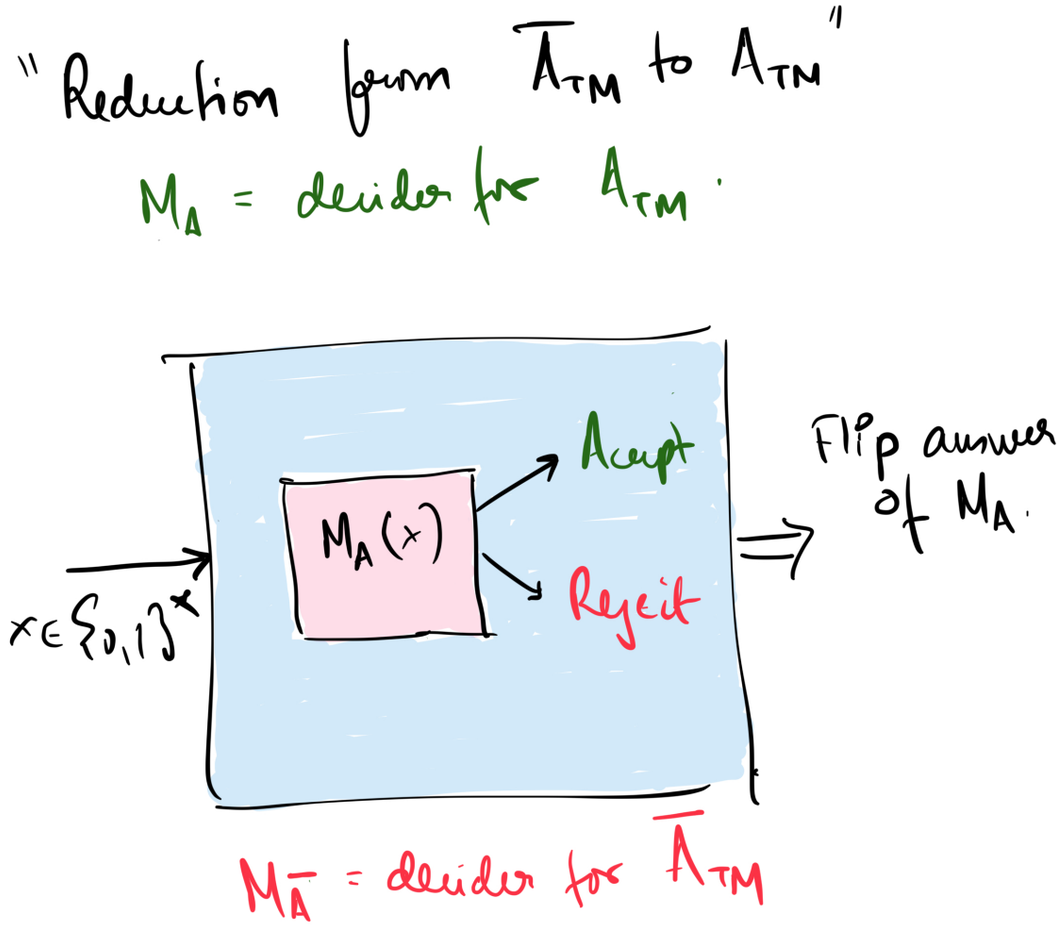
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$A \leq B$ if the algorithm for A has "oracle" access to B



Turing Reducibility vs Mapping Reducibility

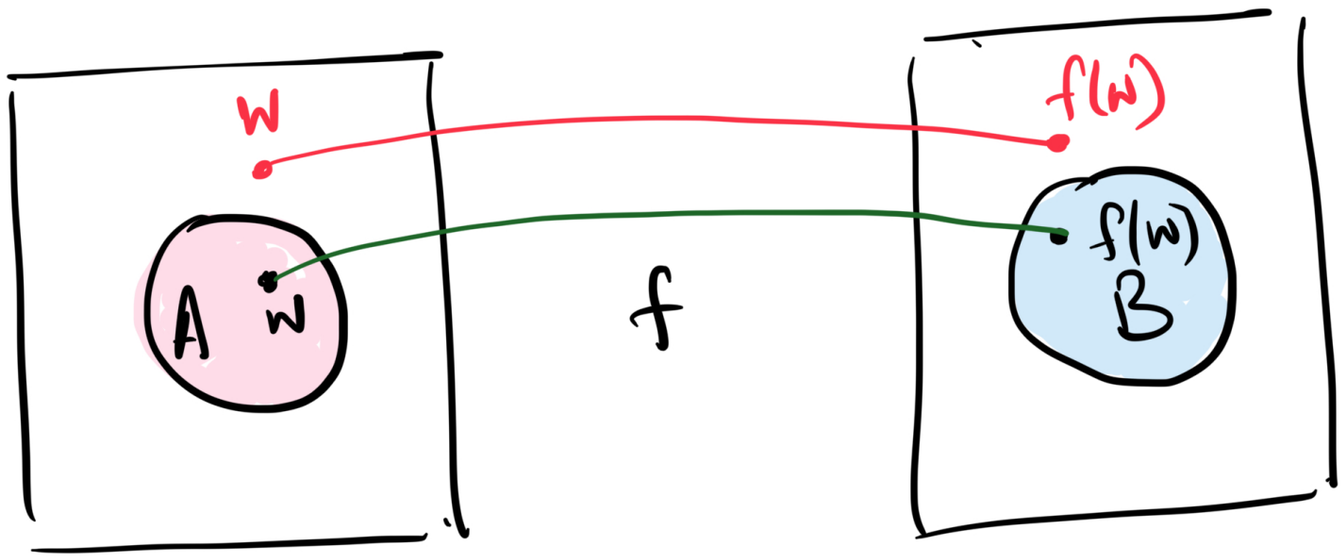
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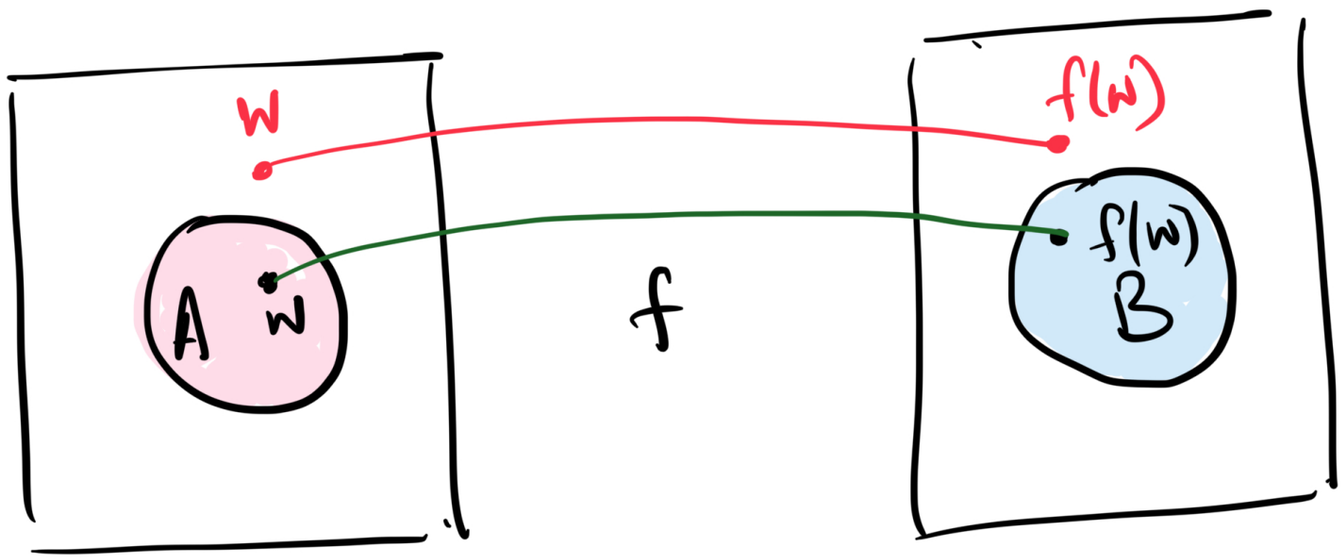
Doesn't let us conclude unrecognizability: $\bar{A}_{TM}$ is unrecognisable, A_{TM} is recognisable.

Mapping Reducibility: More Refined

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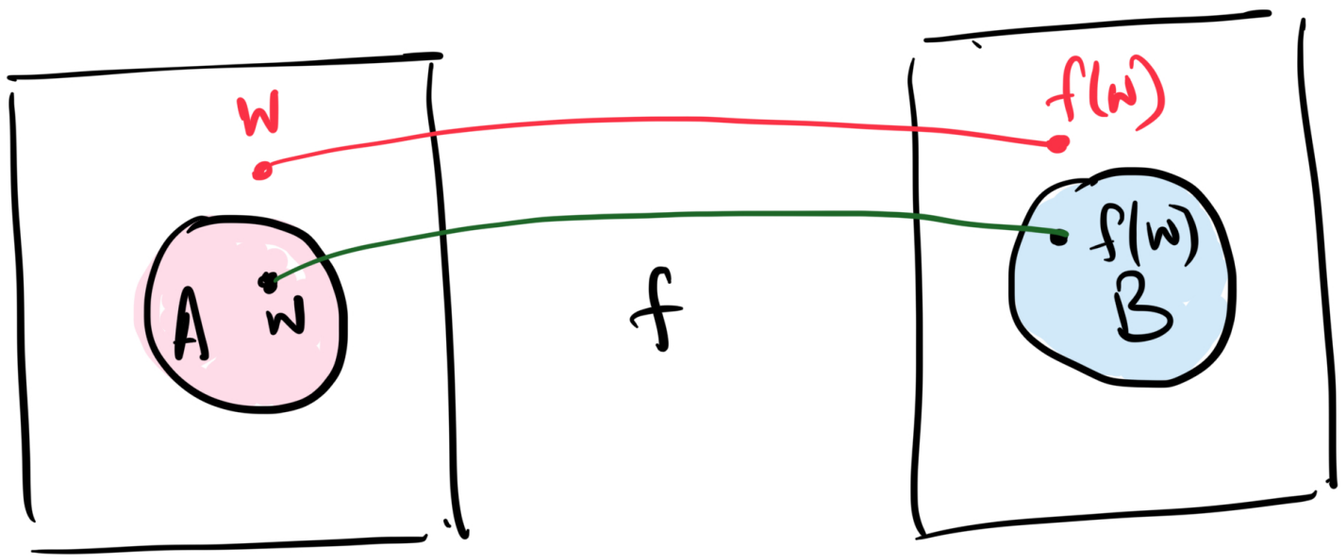


Mapping Reducibility: More Refined



$$A \leq_m B$$

Mapping Reducibility: More Refined



$A \leq_m B$

- Translate from A to B, while preserving membership and non-membership.

- So, we can directly return the answer of the recogniser for B without modifying the answer.

Reducibility Terminology

Why do we use the term "reduce"?

$A \leq_m B$.

- When we reduce A to B, we show how to solve A by using B. We conclude that A is no harder than B. (This suggests the $\leq_m$ notation).
  - Here, reduce means to "boil down a question about A to a question about B".

- **Possibility 1 (Algorithmic perspective):** We bring A's difficulty down to B's difficulty. (Using the solver for B to solve A.)

- **Possibility 2 (Hardness perspective):** We bring B's difficulty up to to A's difficulty. (Showing B is as hard as A.)

Summary

- Introduced the mapping reducibility to show undecidability/ unrecognisability.
- Showed EMPTY$_{TM}$ is undecidable and unrecognisable.
- Showed $\overline{\text{EQ}}_{TM}$ and EQ$_{TM}$ are unrecognisable.
- Showed L_{01} is unrecognisable.