CSC4010: Introduction to Theoretical Computer Science
Problem Set 2

Remark: Each problem mentions which lectures it uses techniques from. If you would like feedback on your homework, you can either submit it in class or email it to both of us. If you find any errors, please let us know!

1. Show the following statements about decidable languages:
   (a) Show that if two languages L_1, L_2 are decidable then so is $L_1 \cap L_2 = \{x \mid x \in L_1, x \in L_2\}$.
   (b) Show that if two languages L_1, L_2 are decidable then so is $L_1 \circ L_2 = \{xy \mid x \in L_1, y \in L_2\}$.

   Note: The most important part of the proof is that you clearly explain what changes to the state set, alphabet, and/or transition function you have to make to prove these statements. You may use a variant of TM discussed in class to prove these.

2. In this course, the ordinary Turing machine has a tape that is infinite in both directions. A Turing machine with a one-way infinite tape is a variant whose tape has a leftmost cell and extends infinitely only to the right. Prove that ordinary (two-way infinite) TMs and one-way infinite TMs decide the same class of languages.

3. Give a Turing Machine that does the following:
   (a) On input 1^k, writes k in binary on the tape and halts. Describe the machine at the level of states and transitions.
   (b) Takes as input the binary representation $\langle x\rangle$ of a non-negative integer x, replaces it by $\langle x+1\rangle$, and halts.

4. In class, we described a Universal Turing Machine. Write a pseudocode describing the behavior of the Universal Turing Machine U on input $\langle M, w\rangle := \langle M\rangle w$, where $\langle M\rangle \in \{0,1\}^*$ is an encoding of a Turing machine M, and $w \in \{0,1\}^*$ is its input. Let $M = (Q, \Sigma, q_{\text{acc}}, q_{\text{rej}}, \delta, q_0, \sqcup)$, where $|Q| = 2^k$ and $|\Sigma| = 2^m$. Recall the following encoding conventions:

   1. $Q = \{0,1\}^k$.
   2. For every $b \in \Sigma$, $\langle b\rangle \in \{0,1\}^m$. In particular, $\langle 0\rangle = 0^m$, $\langle 1\rangle = 10^{m-1}$, $\langle \sqcup\rangle = 1^m$.
   3. $\langle L\rangle = 0$, and $\langle R\rangle = 1$.
   4. $\langle q, b\rangle = q\langle b\rangle$, and $\langle q, b, D\rangle = q\langle b\rangle\langle D\rangle$, for $q \in Q$, $b \in \Sigma$, and $D \in \{L, R\}$.
   5. If $\delta(q, b) = (q', b', D)$, then $\langle \delta(q, b)\rangle = q'\langle b'\rangle\langle D\rangle$.
   6. $\langle \delta\rangle$ is the concatenation of the records $\langle q, b\rangle\langle \delta(q, b)\rangle$ for every $q \in Q$ and $b \in \Sigma$, and
   7. $\langle M\rangle = 1^k 0 1^m 0\langle \delta\rangle$.

   Assume that the encodings of q_0, q_{acc}, and q_{rej} are fixed and known to U. The transition table contains exactly one record for each pair (q, b). Write pseudocode for U that:

   1. decodes $\langle M, w\rangle$;
   2. simulates the transitions of M one at a time; and
   3. accepts, rejects, or runs forever exactly as M does on input w.

   You may assume that the input is a valid encoding. Alternatively, specify what U does on an invalid encoding.