CSC4010: Introduction to Theoretical Computer Science
Problem Set 4

A construction template for unrecognizability proofs

To prove that a language L is unrecognizable, you may construct a mapping reduction $\overline{A_{\text{TM}}} \leq_m L$. That is, define a computable function f satisfying $\langle M, w\rangle \in \overline{A_{\text{TM}}} \iff f(\langle M, w\rangle) \in L$. While the construction of f may seem mysterious, it often follows a template which describe below. Here F refers to the Turing Machine which computes f.

Machine F: On input $\langle M, w\rangle$:

1. Construct a Turing machine N as follows:

   Machine N: On input x:
   (a) **Part 1:** [Initial behavior to be specified.]
   (b) Simulate M on w.
   - If M accepts, proceed to part 2.
   - If M rejects, loop forever.
   - If M loops, continue simulating forever.

   (c) **Part 2:** [Remaining behavior to be specified.]
2. Return $f(\langle M, w\rangle) = \langle N\rangle$.

How to use the template:

- **Part 1** runs before the simulation, regardless of what M does on w. It may accept or reject immediately on selected inputs.
- **Part 2** is reached only if Part 1 continues and M accepts w.
- F **constructs a machine description; it does not run** N**.** Thus, F must halt even when N does not.

Choose Parts 1 and 2 so that:

- If M **does not accept** w (either rejecting or looping), the output belongs to L.
- If M **accepts** w, the output does not belong to L.

Your proof must describe the construction, explain why f is total and computable, establish both cases, and explain why the reduction implies that L is unrecognizable. This template is only a guide. It may not always apply. For example, if the target language concerns multiple machines or a machine-input pair, adapt the construction and its output accordingly. You are also welcome to use other proofs even if it does.

1 Homework Proper

1. Prove the following two statements:
   (a) Show that a language A is recognizable if and only if $A \leq_m A_{TM}$.
   (b) Show that a language A is decidable if and only if $A \leq_m L_{0^*1^*}$, where $L_{0^*1^*} = \{0^m 1^n \mid m, n \geq 0\}$.

2. A **useless** state in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is unrecognizable.

3. Let $L_{\text{finite}} = \{\langle M \rangle \mid M \text{ is a Turing Machine and } L(M) \text{ is finite}\}$, then L_{finite} is unrecognizable.

4. Given two Turing machines M_1 and M_2, we define that M_1 *lags* behind M_2 if the following condition is satisfied: For all input strings $x \in \{0,1\}^*$, if $M_2(x)$ accepts, then $M_1(x)$ loops. (If $M_2(x)$ does not accept, $M_1(x)$ may behave arbitrarily.) Define the language

$$L_{\text{LAG}} = \{\langle M_1 \rangle, \langle M_2 \rangle : M_1 \text{ lags behind } M_2\}.$$

   Prove that L_{LAG} is unrecognizable.