**CSC4010: Introduction to Theoretical Computer Science**
**Problem Set 5**

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**Recap**

- $\mathrm{TIME}(T(n))$ is the class of languages decidable in $O(T(n))$ time, and $\mathsf{P} = \bigcup_{k \geq 1} \mathrm{TIME}(n^k)$.
- A language is in $\mathsf{NP}$ if it has polynomial-length certificates checked by a polynomial-time verifier: every yes-instance has an accepting certificate, and for every no-instance, there is no certificate which will cause the verifier to accept.
- $A \leq_{\mathrm{P}} B$ means that there is a polynomial-time computable function $f$ such that $x \in A \iff f(x) \in B$ for every input $x$.

1. Determine whether each statement is true or false, and give a short explanation.
   (a) If a Turing machine decides a language $L$ with worst-case running time $n^2+10n+100$, then $L \in \mathrm{TIME}(n^3)$.
   (b) If a Turing machine decides a language $L$ with worst-case running time $2^n + 10n + 100$, then $L \notin \mathsf{P}$.
   (c) There exists a decidable language that does not belong to $\mathrm{TIME}(2^n)$.
   (d) For every pair of functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \log f(n) = o(g(n))$ and $g$ is computable, $\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$.
   (e) If $A \leq_{\mathrm{P}} B$ and $A \in \mathsf{P}$, then $B \in \mathsf{P}$.

2. Consider the language
$$L = \big\{w \in \{0,1\}^* : |w| = 2^k \text{ for some integer } k \geq 0\big\}.$$
   (a) Give a high-level description of a Turing machine that decides $L$. You may use either a single tape or multiple tapes.
   (b) Determine a tight asymptotic bound on the worst-case running time of your machine, using $\Theta$-notation.

3. Let $G = (V, E)$ be a finite, simple, undirected graph. An *independent set* is a set of vertices in which no pair of distinct vertices is joined by an edge.

   For positive integers $k$, define
$$\text{INDEPENDENT-SET} = \big\{\langle G, k\rangle : G \text{ has an independent set of size } k\big\}.$$
   (a) Show that $\text{INDEPENDENT-SET} \in \mathsf{NP}$ by describing certificates and a polynomial-time verifier. Explain why your verifier is correct and why the certificate length and running time are polynomial in the input length.
   (b) Show that $\text{INDEPENDENT-SET} \in \mathsf{NP}$ using a polynomial-time reduction to CLIQUE. Describe the reduction, prove its correctness, and explain why it is computable in polynomial time and establishes the desired conclusion. You may use the fact that $\text{CLIQUE} \in \mathsf{NP}$.