Ruling Sets in Random Order and Adversarial Streams

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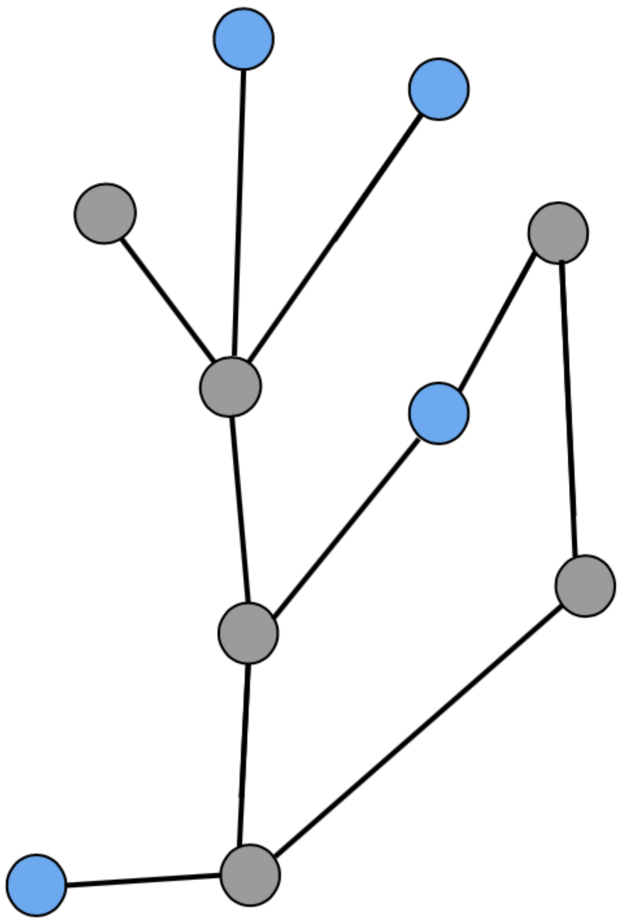
Ruling Sets

(α, β)-Ruling Sets:

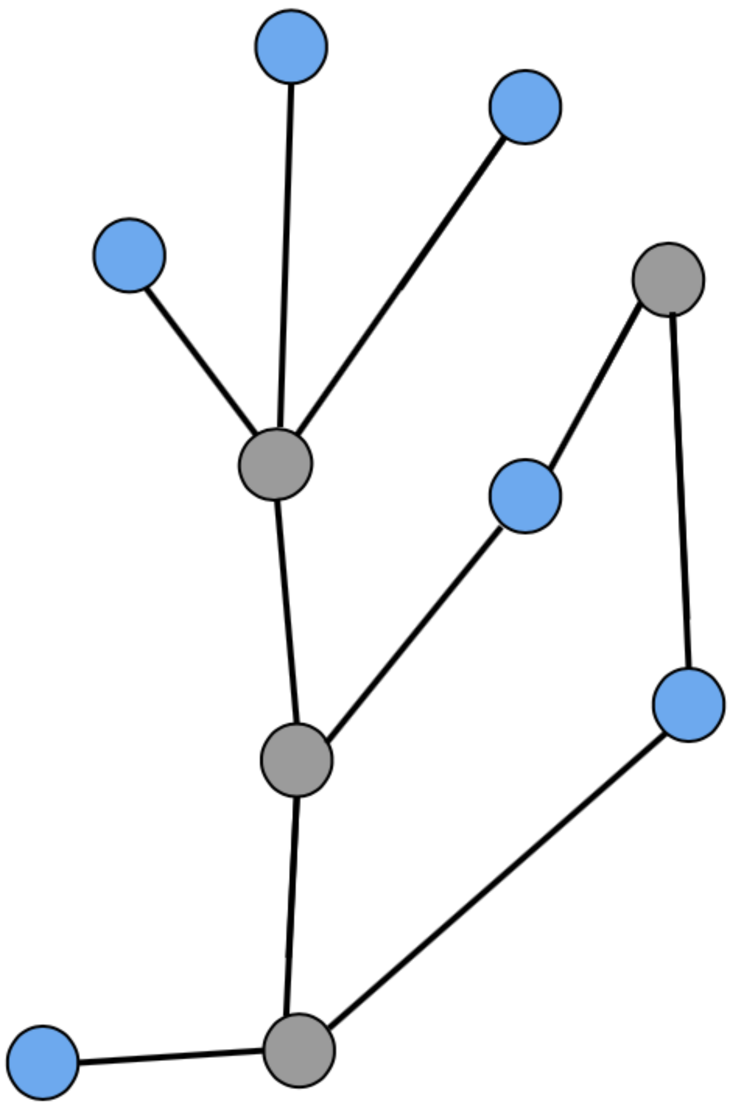
- The distance between each pair of vertices in the ruling set is at least α .
- Each node not in the ruling set is at a distance at most β from some node in the ruling set.



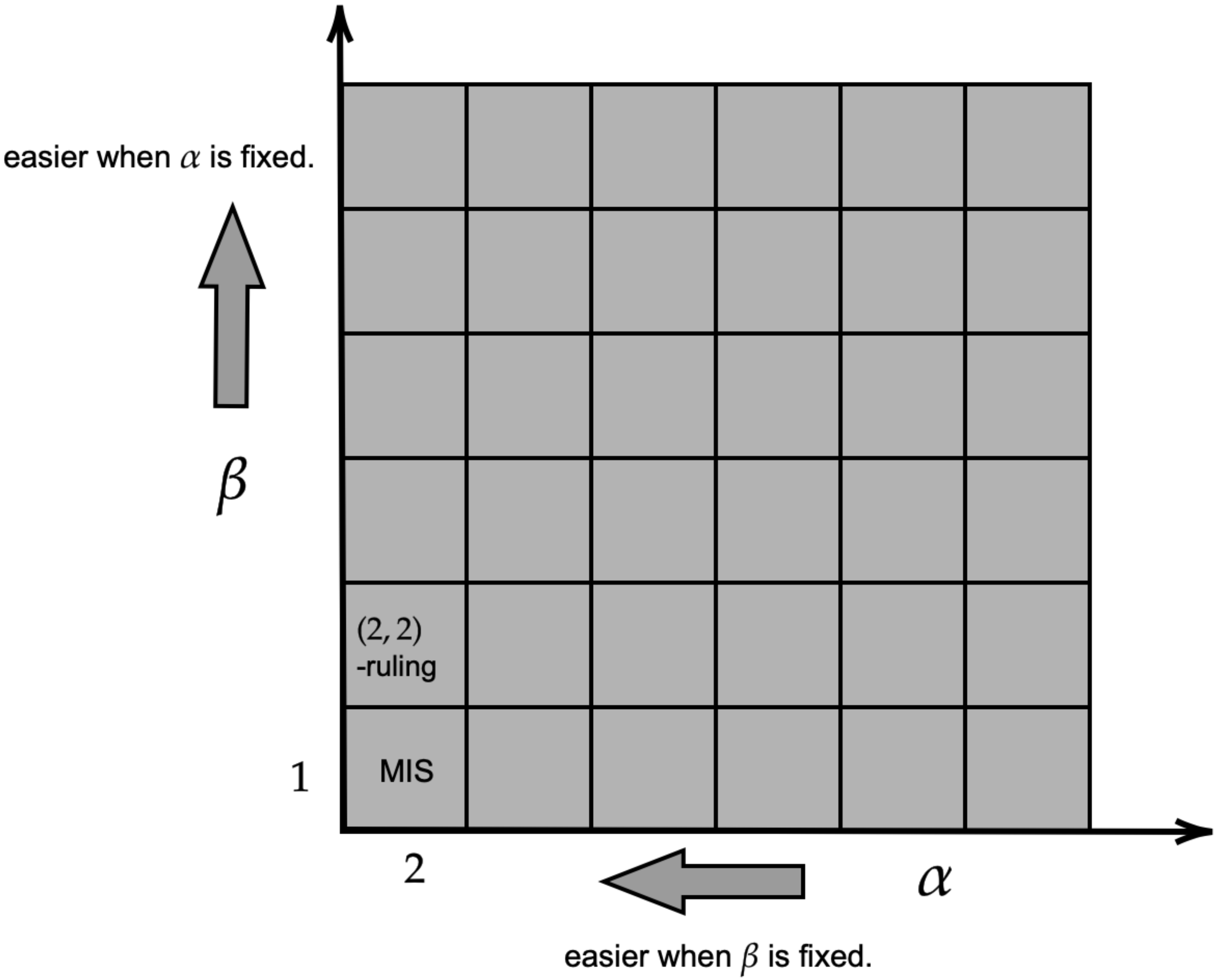
(2,1)-Ruling Set = MIS

- **Independent Set:** The vertices of the set aren't adjacent to each other.

- **Maximality:** We cannot add vertices without violating independence.



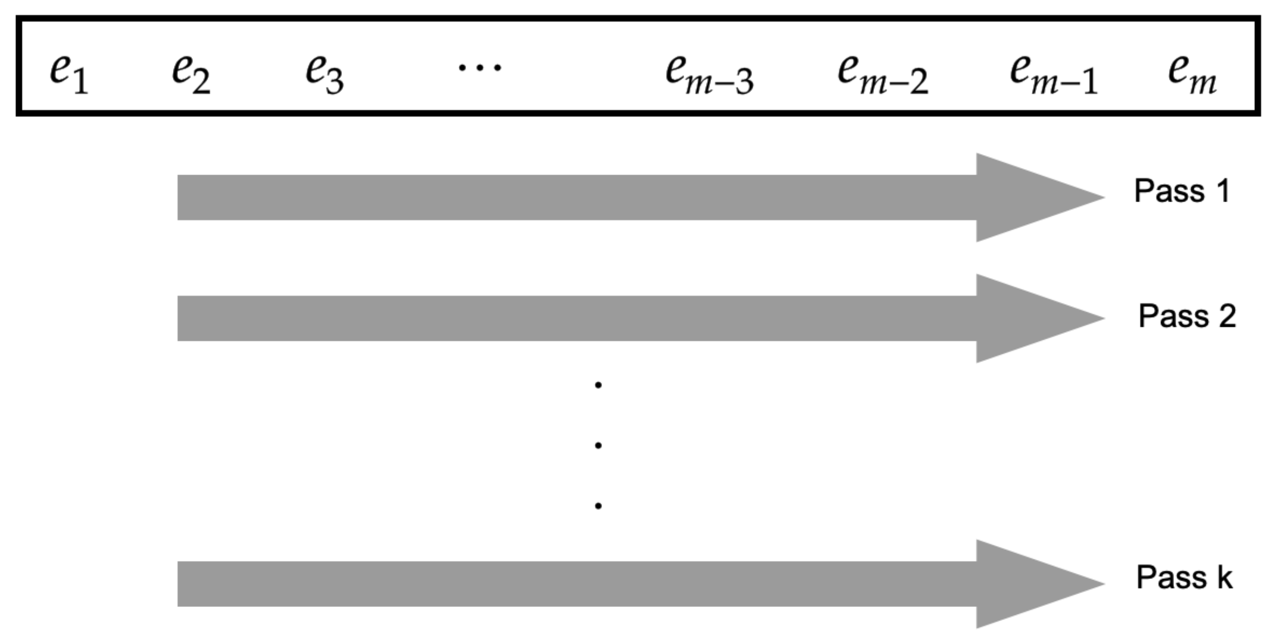
Ruling Sets


easier when α is fixed.
β
$(2,2)$
-ruling
MIS
1
2
α
easier when β is fixed.

Graph Streaming

Graph $G = (V, E)$:

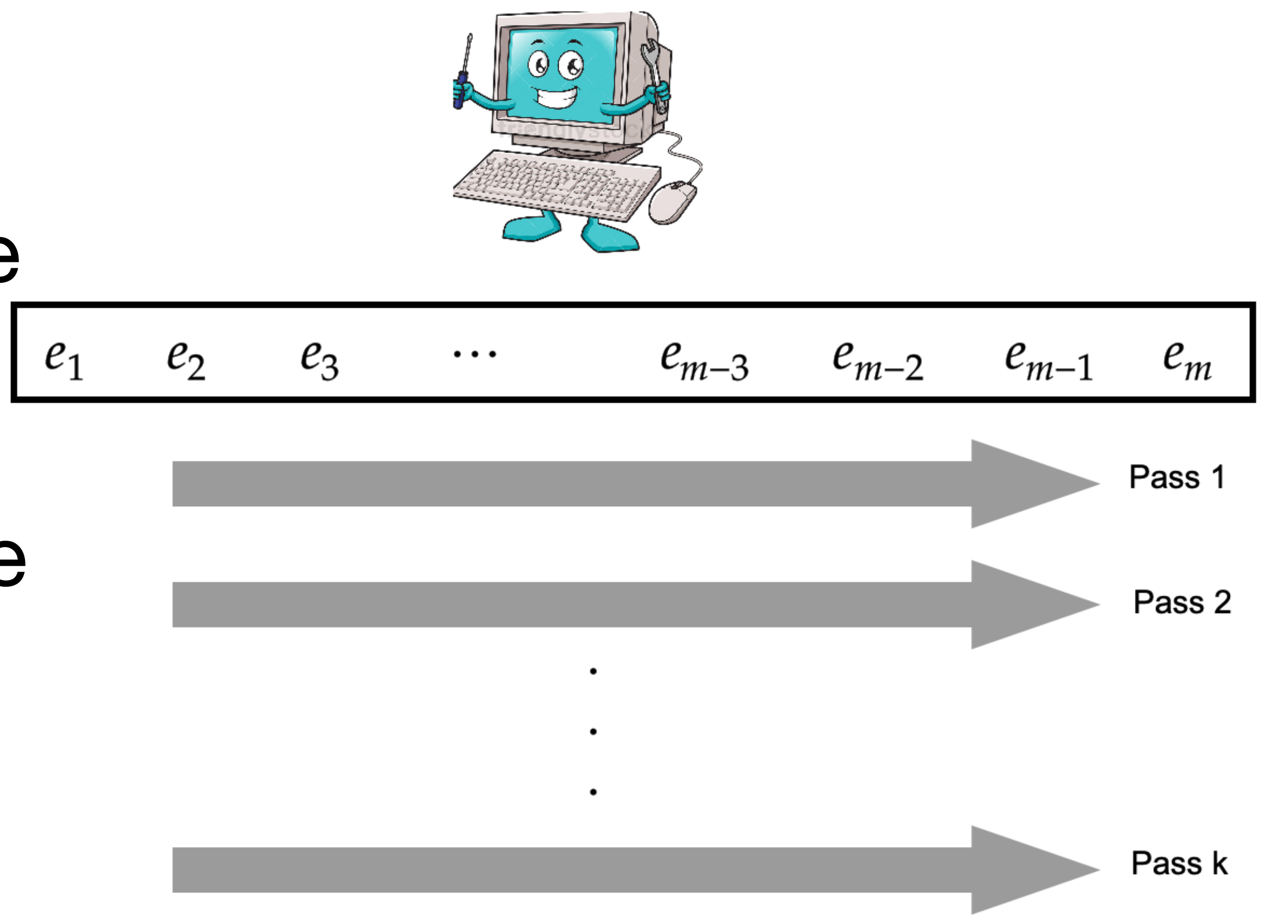
- Known vertices:
  $V = \{v_1, v_2, \cdots, v_n\}$
- Unknown edges:
  $E = \langle e_1, e_2, \cdots, e_m \rangle$



Random-Order Streams

- The adversary can choose the graph.

- The edges $\langle e_1, e_2, \cdots, e_m \rangle$ arrive in a **random order**.



Results

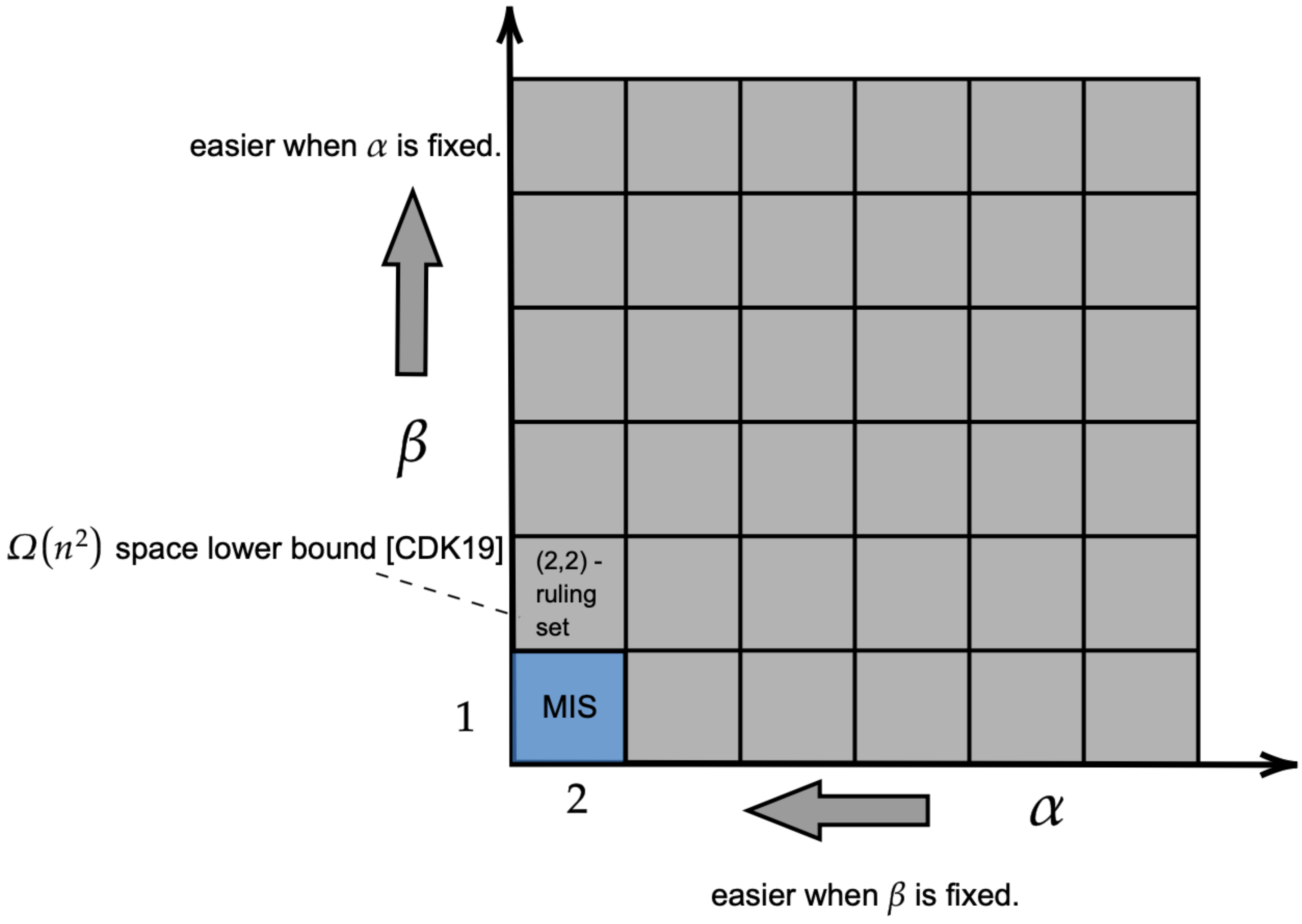
Random-Order Streams:

- An $\tilde{O}(n)$ - space streaming algorithm for $(2,2)$ - ruling sets.

Adversarial Streams:

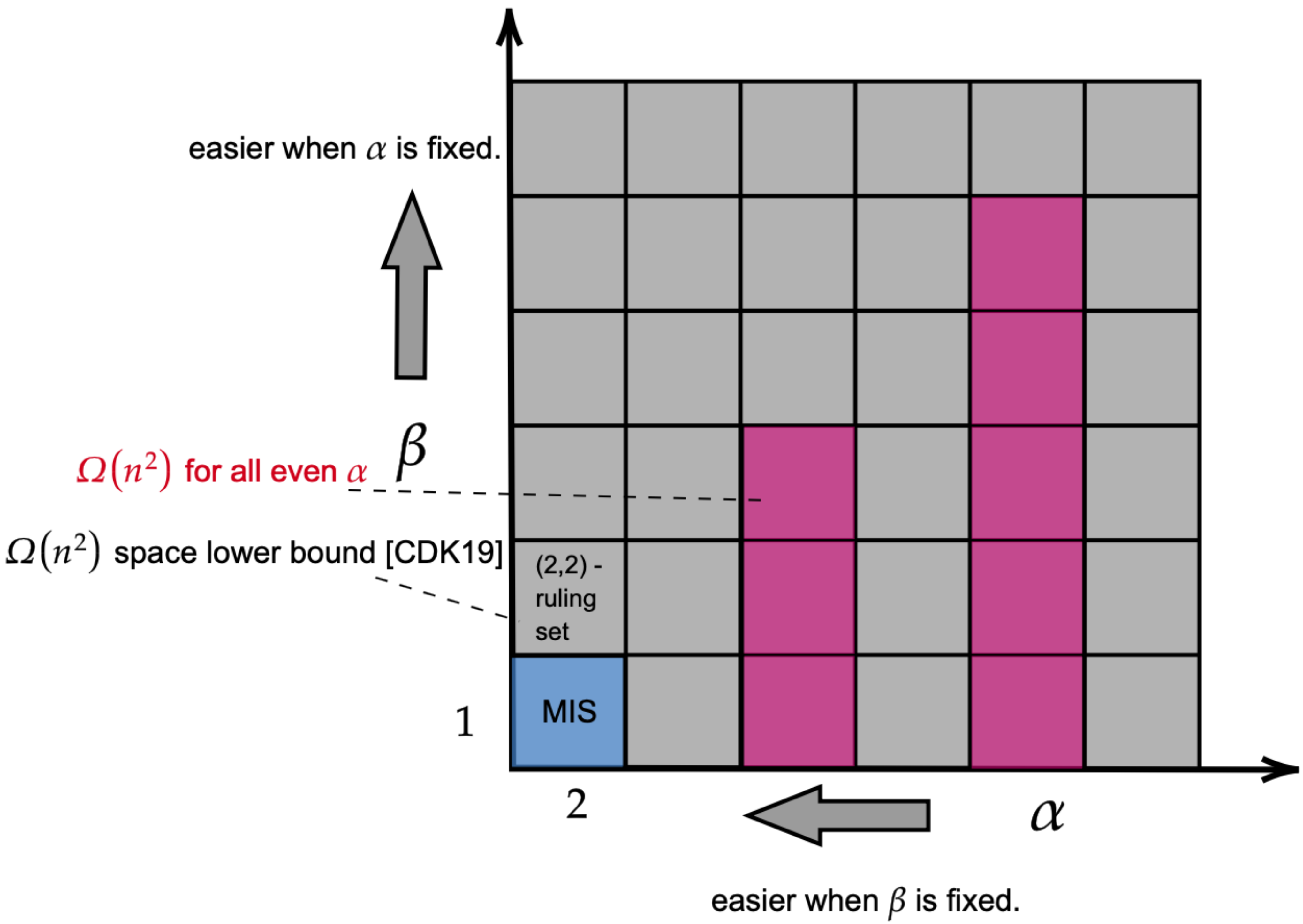
- An $\tilde{O}(n^{4/3})$ - space streaming algorithm for $(2,2)$ - ruling sets.
- An $\Omega(n^2)$ space lower bound for any streaming algorithm computing a $(\alpha, \alpha - 1)$ ruling set for <u>even</u> α .

Results



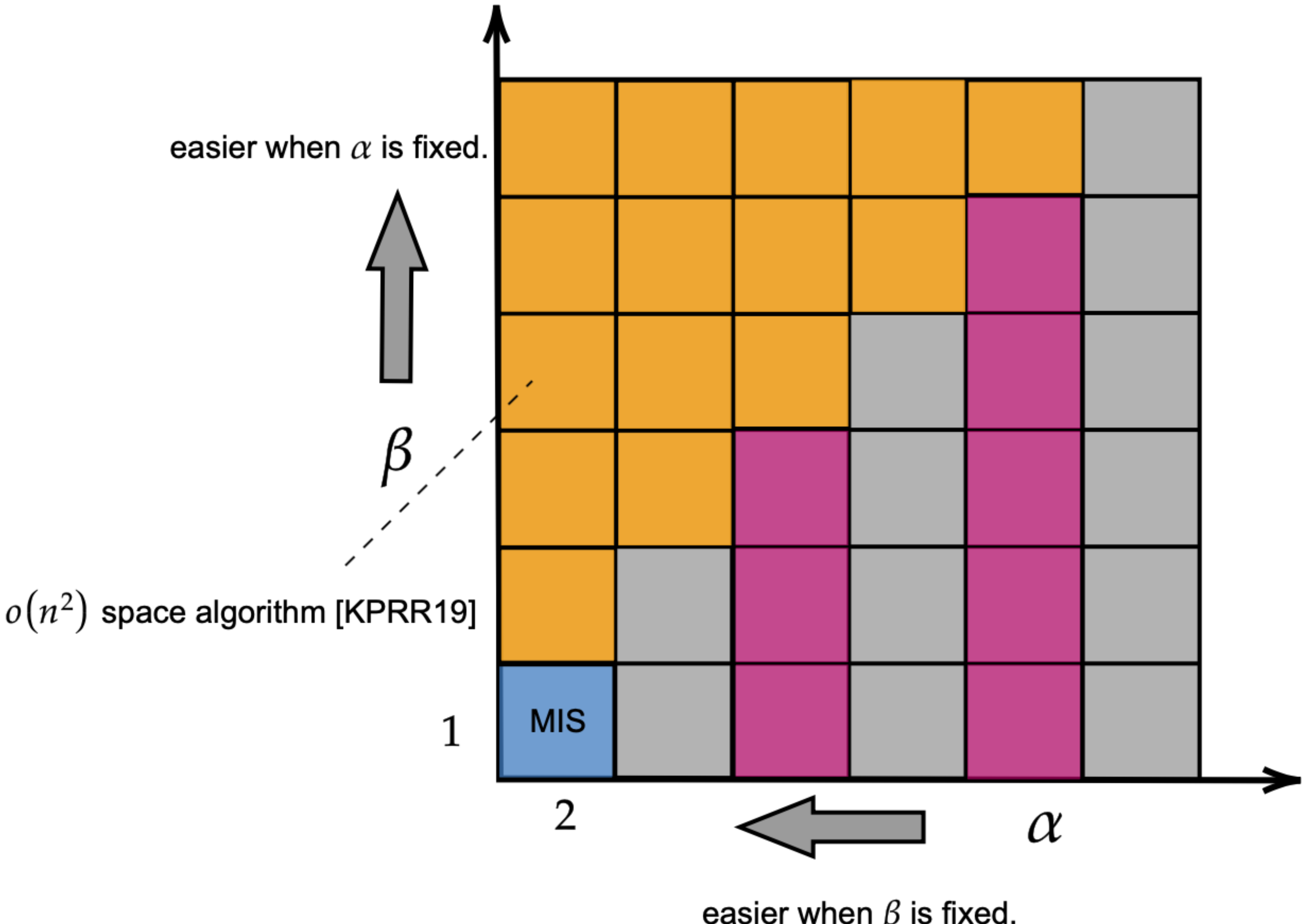
easier when α is fixed.
β
$\Omega(n^2)$ space lower bound [CDK19]
(2,2) - ruling set
MIS
1
2
α
easier when β is fixed.

Results



easier when α is fixed.
β
$\Omega(n^2)$ for all even α
$\Omega(n^2)$ space lower bound [CDK19]
(2,2) - ruling set
MIS
1
2
α
easier when β is fixed.

Results


easier when α is fixed.
β
$o(n^2)$ space algorithm [KPRR19]
MIS
1
2
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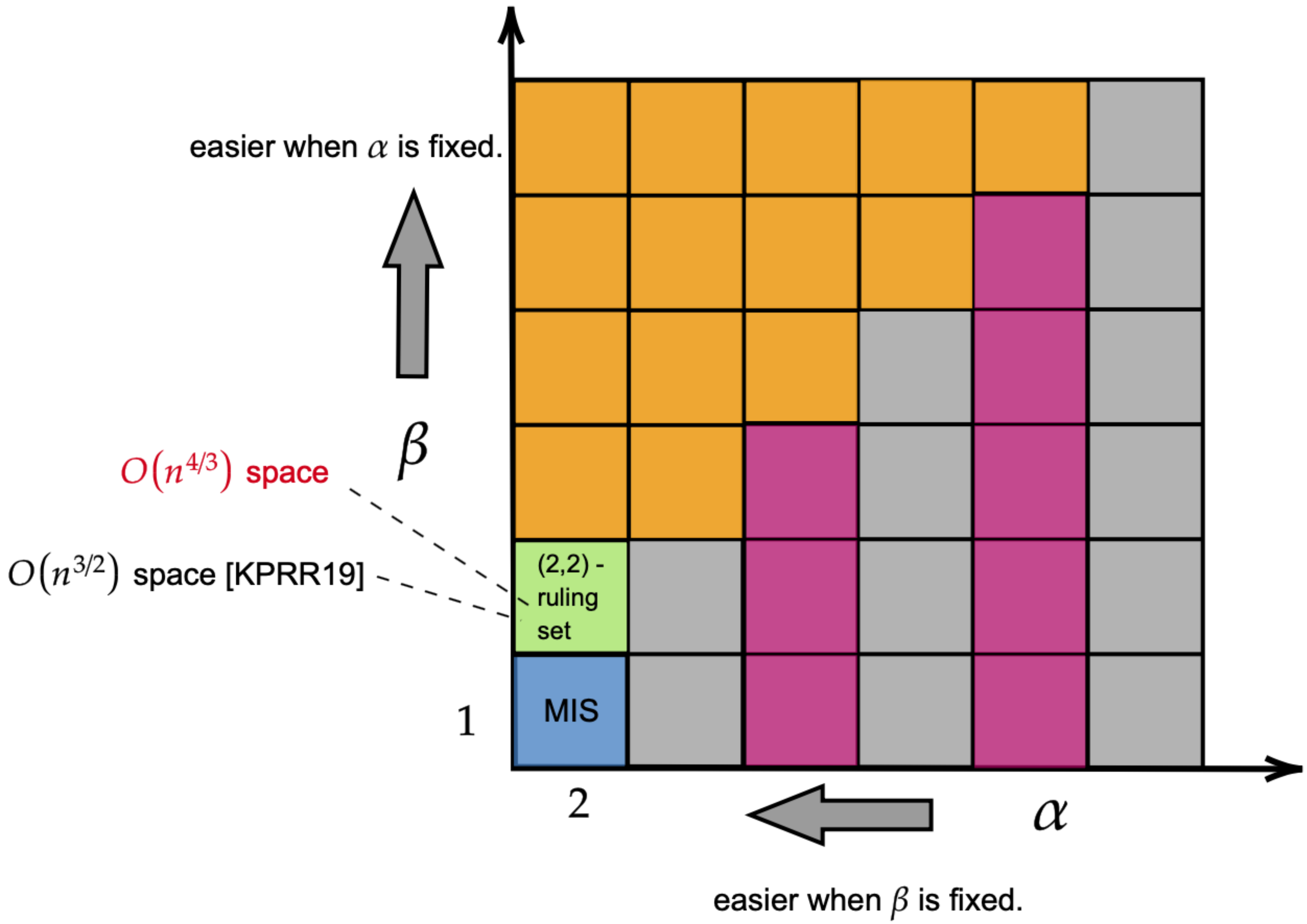
easier when β is fixed.

Results



easier when α is fixed.
β
$O(n^{4/3})$ space
$O(n^{3/2})$ space [KPRR19]
(2,2) - ruling set
MIS
1
2
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easier when β is fixed.

Streaming Algorithm

- Starting Point: Decomposition due to [KP12], [BKP14].

$$\textbf{Let } r = \log n - \log\log n,\ d_o = n, d_i = \frac{n}{2^i} \textbf{ for } i \in [r]$$

$$V_0 = V, E_0 = E,\ \textbf{and } G_0 = G\,.$$

$$\textbf{For } i \geq 1, V_i = \{v \in V_{i-1} \mid \mathbf{deg}_{G_{i-1}}(v) \leq d_i\}$$

$$G_i = G[V_i], E_i = E(G_i)$$

Streaming Algorithm

1. **For each** $i \in [r-1]$, **sample** S_i **of size** $\dfrac{10\,|V_i|\log n}{d_{i+1}}$ **from** V_i.

2. **Let** $H = G[\cup_{i=1}^{r-1} S_i \cup V_r]$. **Output MIS of** H.

- Claim: H is a (2,2)-ruling set of G with high probability.

  Proof: Each $v \in V_i \backslash V_{i+1}$ has a neighbor in S_i.

Streaming Algorithm

- Sets V_i for $i \in \{1,2,\cdots,r\}$ are unknown and are (possibly) hard to determine in adversarial streams.
- But easier in random order streams!

Streaming Algorithm

- Since edges arrive in a random order, we can estimate degrees by looking at a small part of the stream.

$$\underbrace{e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_{100n\log n}} \quad \cdots \quad e_{m-3} \quad e_{m-2} \quad e_{m-1} \quad e_m$$

Let $T_1 = \{e_1, e_2, \cdots, e_{100n\log n}\}$. If $deg_{T_1}(v) \leq 50\log n$, add v to $\widetilde{V_1}$.

- Claim: **For** $v \in \tilde{V}_1$, **deg**$(v) \leq \dfrac{n}{2}$.

Streaming Algorithm

- Keep repeating:

$$\underbrace{e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_{100m\log n/d_{i+1}}} \quad \cdots \quad e_{m-3} \quad e_{m-2} \quad e_{m-1} \quad e_m$$

Let $T_{i+1} = \{e_1, e_2, \cdots, e_{100n\log n/d_{i+1}}\}$. If $deg_{T_{i+1}\cap G\left[\widetilde{V_i}\right]}(v) \leq 50\log n$, add v to $\widetilde{V_{i+1}}$.

- Claim: **For** $v \in \tilde{V}_{i+1}$, $\mathbf{deg}_{\tilde{V}_i}(v) \leq \dfrac{d_i}{2}$. **For** $v \in \tilde{V}_i \backslash \tilde{V}_{i+1}$, $\mathbf{deg}_{\tilde{V}_i}(v) \geq d_i$.

Open Questions

- Complexity of MIS in random order streams?

- Is there a lower bound for (2,2)-ruling sets in adversarial streams? Can we get a better upper bound?